EQUATION OF STATE AT ULTRAHIGH DENSITIES

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PART 2

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Ah, but a man's reach should exceed his grasp, or what's a heaven for?

Browning

1 INTRODUCTION

In Part 1 of this review (Canuto 1974) we analyzed the present knowledge about the behavior of matter at densities up to a few times nuclear density. We concluded by presenting in Tables 5 and 10 the best relations $P = P(\varepsilon)$ for a system of pure neutrons in the region $10^4 \leqq \rho \leqq 2 \times 10^{14}$ g cm^{-3}. In this second part we review the work done so far in the high density region $\rho \geqq 2 \times 10^{14}$ g cm^{-3}.

First, we review the work concerning the appearance of hyperons at densities higher than nuclear density and their influence on the equation of state. Second, we review the work concerning the possible existence of a solid neutron core. The hypothesis that a liquid of neutrons could solidify at high enough densities has recently been the subject of several microscopic detailed computations; the present state of the art is discussed. In Section 4 we discuss several (so far) unrelated attempts to describe the behavior of matter in the relativistic region, that is, at densities greater than 10^{16} g cm^{-3}. All the studies of this region published so far deal with a relativistic liquid of neutrons; should the neutrons indeed solidify at lower densities, the previous relativistic treatment ought to be modified.

One of the most disturbing features of the superhigh-density region is the disagreement in the predicted value for the velocity of sound c_s (in units of c) in the relation $P = c_s^2 \varepsilon$. Some theories predict $c_s^2 \to 1$, whereas others predict $c_s^2 \to \frac{1}{3}$ (free particles) or even lower values.

The available data regarding neutron stars, in particular the moment of inertia, are shown in Section 5 to be insufficient to pin down a specific value of the velocity of sound at superhigh densities. A possible answer is found, however, by analyzing the multiparticle production in the high-energy proton-proton (p-p)

scattering. The Landau hydrodynamic model is reviewed and it is found that the value $c_s^2 \to 1$ seems favored, thus providing a way of continuing the equation of state into the superhigh-density region.

2 THE HYPERONIC LIQUID

2.1 *Generalities*

After the clusters of neutrons and protons that form the crust of a neutron star have dissolved at a density around 10^{14} g cm^{-3}, the system is left in a state composed predominantly of neutrons. If one neglects the possible existence of other particles, like hyperons, the treatment of a pure neutron fluid has already been presented in Section 5 of Part 1, with the corresponding equation of state given in Table 10 (1). If the treatment of the region $\rho > 2 \times 10^{14}$ g cm^{-3} as a fluid of pure neutrons is a good approximation, it is nevertheless true that physically the situation is much more complex, due to the energetically favorable appearance of hyperons. Before going into any detail, it is important to summarize the results by saying that all the computations of the hyperonic liquid published so far have reached the same conclusion, that is, the equation of state is not severely altered from the one corresponding to a pure neutron gas. This statement has to be taken with extreme care. The fact that almost all the computations reach the same conclusion does not make that conclusion necessarily right. In fact, all the computations share the same defect: the hyperonic potentials are taken to be almost identical to the nucleon-nucleon (NN) case. Should future studies of the hyperonic forces reveal unexpected features, the equation of state could be drastically changed. From the historical point of view the first to recognize that hyperons could be present were Cameron (1959) and Ambartsumyan & Saakyan (1960). Salpeter (1960) concluded on general grounds that Σ^- should appear first, a conclusion confirmed by all the detailed computations published since.

2.2 *Formulation of the Problem*

Consider a system composed of k different species, characterized by a concentration n_k. We want to find n_k subject to the condition that the total number of baryons is constant and that the total charge of the system is zero. The problem is solved by evaluating a simultaneous set of chemical potential equations obtained by minimizing the Gibbs free energy. For instance, for a system of e^-, n, p, Λ, Σ, these equations read

$$\begin{aligned} &\mu_n = \mu_{e^-} + \mu_p, \qquad \mu_{\Sigma^-} = \mu_n + \mu_{e^-}, \\ &\mu_\Lambda = \mu_n, \qquad \mu_{\Sigma^0} = \mu_n, \qquad \mu_{\Sigma^+} = \mu_n - \mu_{e^-}. \end{aligned} \tag{2.1}$$

We can easily find some interesting results even without knowing the exact form of the μ's. For example, at a density of 2.39×10^{14} g cm^{-3}, $\mu_e = 103.57$ MeV, $\mu_n = 20.37$ MeV $+ m_n c^2$, and $\mu_p = -81.92$ MeV $+ m_p c^2$ [Table 2 (Part 1)]. The first relation (2.1) is satisfied. If there is no interaction among the particles, the minimum value of μ is just the rest mass. At 2.39×10^{14} g cm^{-3}, Λ cannot exist as yet, because $\mu_\Lambda \approx m_\Lambda c^2 = 1115$ MeV; even Σ^- cannot be present either;

in fact, $\mu_n+\mu_e \approx 124$ MeV $+m_n c^2$ is smaller than $m_\Sigma \sim 1190$ MeV; even worse is the situation for Σ^0 and Σ^+. Admittedly, these conclusions can be significantly altered if account is taken of the potentials acting among the particles.

2.3 *The Multicomponent Many-Body Theory*

If we want to treat the particles in a realistic model, their attraction and repulsion must be included in the computation. Two difficult problems immediately arise:

1. The two-body potential between hadrons is considered known as long as we deal with neutrons and protons. In fact, one has at hand many good potentials, among which one of the favorites nowadays is the one constructed by Reid
2. The situation is very different for any two hyperons or even for the hyperon-nucleon (Y-N) case. The experimental information is still insufficient to determine the hyperonic potentials with the same degree of reliability as the nucleon-nucleon case. No high-energy data exist on YY or YN scattering to allow one to fix the parameters of the potentials in the high angular momentum waves. No experimental information exists, for instance, about $\Sigma\Sigma$ or $\Lambda\Lambda$ for the P waves, which, however, are of primary importance at high densities

The simplest possible choice for the hyperonic potentials consists in assuming that they are the same as the NN case, if one excludes the one pion contribution.

Reasonable though it could sound, this procedure is by no means entirely satisfactory, because it is known (Brown, Downs & Iddings 1970) from ΛN cross-section and singlet (triplet) scattering lengths that much better results are obtained if the ΛN scattering is considered to go through an intermediate state ΣN: pions can then be exchanged. Besides, between two Σ's, new mesons can be exchanged that are not contained in Reid's potential, for example, the K meson with $T = \frac{1}{2}$. An explicit example of how misleading this procedure could be will be given in 2.4.4.

To improve upon this method, one has to employ a full meson-exchange hyperonic potential, which, in principle, can be derived by the exchange of mesons. Evidently the coupling constants entering into the final form of V_{12} cannot be determined with the same accuracy as in the NN case. This point is discussed in more detail below. For the time being, we proceed with the presentation of the many-body formalism as if the potentials were indeed known. When the particles interact with one another the general expression for μ is

$$\mu = (c^2p^2+m^2c^4)^{1/2}+U(p), \qquad 2.2$$

where $U(p)$ is the one-body potential felt by the particle with Fermi momentum p. From Brueckner's many-body theory (Brueckner et al 1968), we know that the one-body potential felt by the particle i is given by

$$(2\pi)^2 U(p_i) = \int_0^{p_i} p_j^2\, dp_j [K(p_i/p_j)-\text{exch}] + \sum_{l\neq i}^{n_k} \int_0^{p_l} p_j^2\, dp_j\, K(p_i/p_j). \qquad 2.3$$

In the first integral we integrate over the particle j, which we take to be of the same kind as i. The second integral refers to the interaction of i with a second particle j, when j is of a different nature. Because the particles are then distinguishable, the Pauli principle does not apply and therefore there is no exchange. Each integral over p_j runs up to p_l, where p_l is the Fermi momentum of the species l. Evidently, the various Fermi momenta are the unknown of the problem, since they are related to the concentrations.

The K-matrix is proportional to the matrix element of the two-body potential, taken between the perturbed and the unperturbed wave functions. The perturbed wave function is in turn the solution of an integral equation (Schrödinger equation with the appropriate boundary conditions), whose Green function is again given

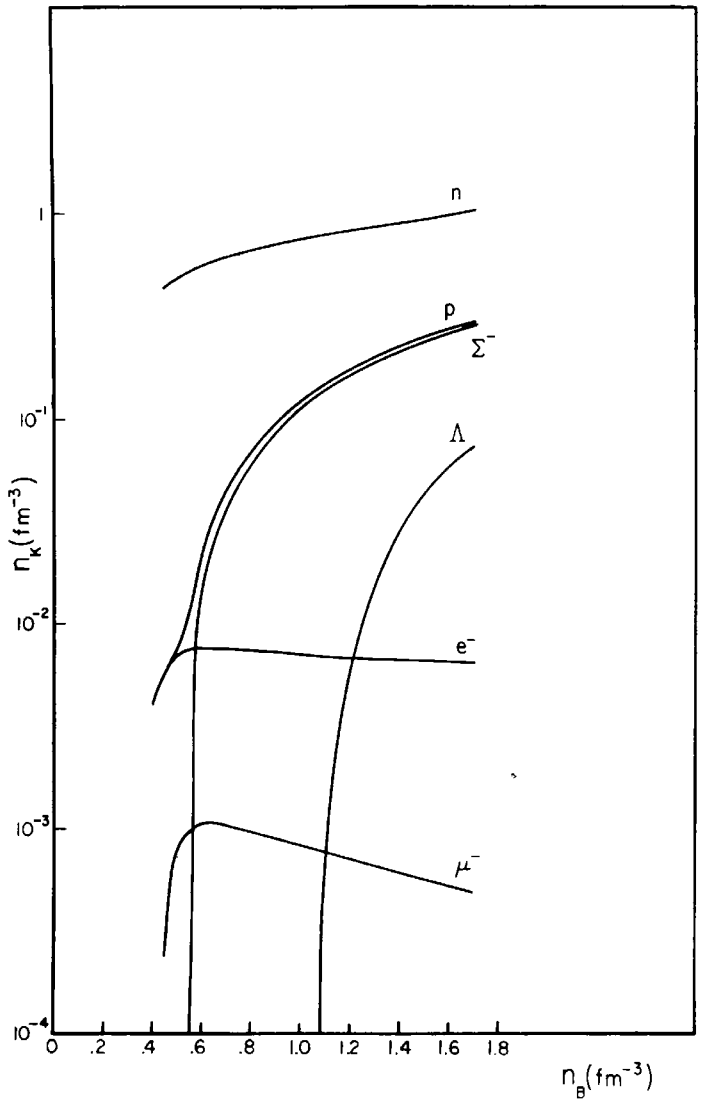


Figure 1 Concentrations of a free hyperonic liquid vs baryonic density, as from the work of Ambartsumyan & Saakyan (1960).

in terms of the one-body potential $U(p)$. This self-consistency is the most important feature of the Brueckner many-body theory.

If it is difficult to handle such a feature in the nuclear matter case where there are only neutrons and protons, it is clear that it becomes a tremendously involved problem when we want to deal with say 8 or 10 different species. Unfortunately, none of the results for the hyperonic liquid published so far has treated this essential feature in a satisfactory way.

2.4 *Review of the Results*

2.4.1 THE WORK OF AMBARTSUMYAN & SAAKYAN (1960) When the particles are noninteracting, the system 2.1 can be solved almost analytically. Ambartsumyan & Saakyan (1960) solved such a system by considering n_n as the independent variable, instead of the customary baryonic number density, n_B. We have repeated the computation by solving the system of equation 2.1 as a function of n_B, in order to make the future presentation of other results easier. We have considered eight variables: e^-, μ^-, n, p, Λ, Σ^0, Σ^-, and Σ^+. In the absence of interaction the chemical potentials are given by 2.2, with $U(p) = 0$.

The system 2.1 must be solved simultaneously with the constraint of charge and baryonic number conservation

$$e^- + \mu^- + \Sigma^- = p + \Sigma^+, \qquad n + p + \Lambda + \Sigma = n_B, \tag{2.4}$$

where the symbol of a particle stands for its numerical density. The results of Ambartsumyan and Saakyan are shown in Figure 1.

2.4.2 THE WORK OF LANGER & ROSEN (1970) The Levinger-Simmons NN nonlocal potential, equation 3.9 (1), was adopted for any two hadrons. We do not know at present if this assumption is correct. The hyperon-nucleon data at low energy indicate that the YN potentials are rather different from the NN potentials, as indicated in Figure 3. The authors considered the following particles: n, p, e^-, μ^-, Σ^-, Σ^0, Σ^+, Δ^-, and Δ^0.

The treatment was based on a Hartree-Fock definition of the one-body potential $U(p)$; no self-consistency requirement was employed nor was a multicomponent many-body theory as outlined before. The results are shown in Figure 2. The first hyperon to enter is indeed Σ^-, and its appearance is accompanied by a decrease in the number of electrons. This is clearly due to charge conservation, because one can use Σ^- to counterbalance the proton charge. Σ^- are more advantageous than e^-, which are rather expensive due to their small mass: their energy goes up rather fast with density, because they lack any attractive interaction to lower the energy.

The Levinger-Simmons potential is given in two different versions, the so-called V_α and V_β potentials, that differ slightly in their analytic behavior. The results presented here correspond to the choice of V_α, the alternative choice giving rather similar results.

As we said before, these results can substantially change if the potentials are treated more realistically and the many-body effects are included.

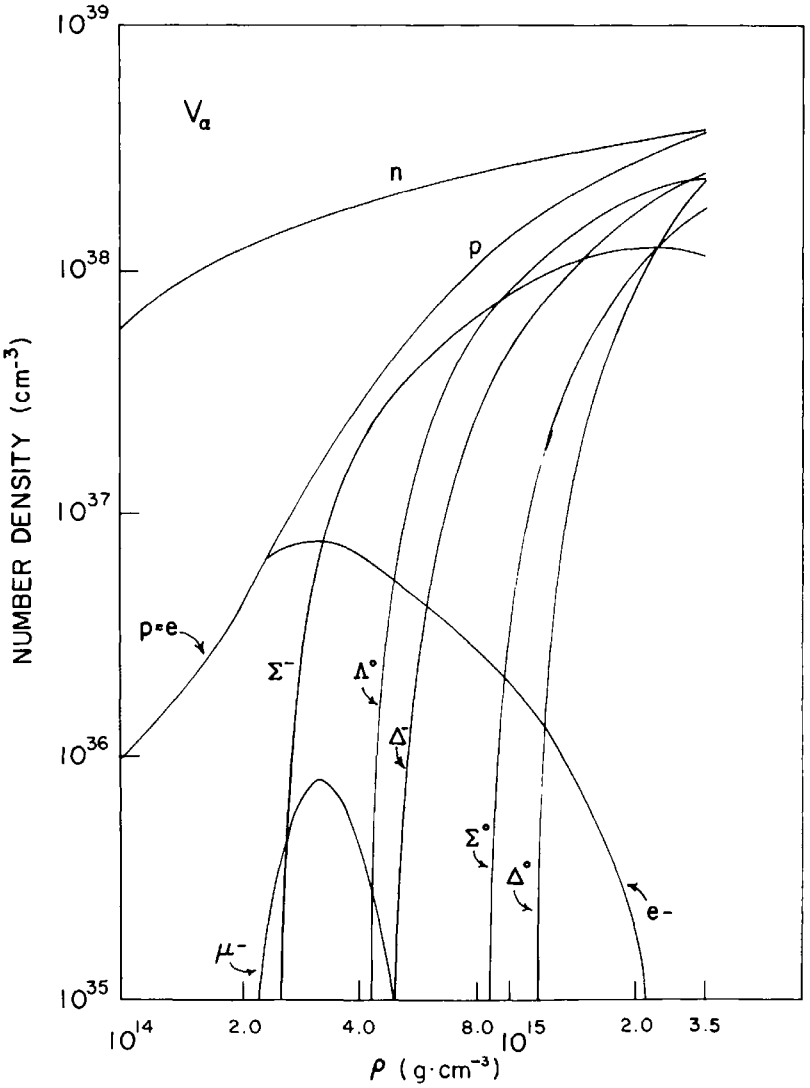


Figure 2 Hyperonic concentrations vs matter density as from the work of Langer & Rosen (1970).

2.4.3 THE WORK OF BUCHLER & INGBER (1971) Were it not for the limited number of particles employed, the work of Buchler & Ingber (1971) would be a considerable improvement upon the previous work, in that it is based on a self-consistent many-body calculation with realistic potentials. The two-body potential used was the one previously derived by Ingber & Potenza (1970), based upon a phenomenological meson-nucleon Lagrangian. It is nonlocal, that is, it contains momentum-dependent terms and has a vanishing tensor force at short distances. The potential was checked against nucleon-nucleon phase shifts, the quadrupole moment of the deuteron and binding energy of nuclear matter. The authors included the following particles: n, p, e^-, μ^-, and π^-, and performed the calculation in the density region $2.5 \times 10^{37} \leqq n_B \leqq 3.55 \times 10^{38}$ cm^{-3}. Their results were presented in Figure 12 of

Part 1 because they constitute a continuation of the structure after the coalescence of the nuclei of the crust.

The flattening of the concentration of μ^- at $n_B = 2.66 \times 10^{38}$ cm^{-3} is entirely due to the presence of pions. However, the pions are not treated correctly because their strong interaction with nucleons, repulsive in S-state and attractive in P-state, has been ignored.

The real progress with respect to the previous work lies in the self-consistency requirement; the concentration of each species depends on each of the other species and not only on the average matter density.

2.4.4 THE WORK OF PANDHARIPANDE & GARDE (1971, 1972) In a first paper, Pandharipande (1971) studied the composition of a baryonic liquid using a many-body theory based upon Jastrow's variational approach: the two-body wave function was taken as satisfying a simplified form of the Bethe-Goldstone equation, in which the complicated inhomogeneous term, representing the Pauli principle, was omitted but simulated by imposing a healing condition. Instead of a state-dependent correlation function, an average was used and only its spin dependence (singlet or triplet) was introduced.

The one-body potential $U(p)$ was assumed to depend only upon the total density. Because it has been stressed that the density dependence of $U(p)$ is a complicated function of the concentration of each species, the approximation used in this work is far from exhaustive.

Three models were studied with different choices for the constituents and the potentials. In particular:

A $n, p, \Lambda, \Sigma, \Delta, e^-, \mu^-$

B $n, p, \Lambda, \Sigma, e^-, \mu^-$

C $n, p, \Lambda, \Sigma, \Delta, e^-, \mu^-$.

In A, the same interaction was used for any two baryons. In B, the Δ hyperons were excluded, but otherwise the potential was the same as in A. The potentials were assumed to be purely central and, in particular, between two hyperons or a hyperon and nucleon they were taken to be ($x = 0.7r$).

$$l = 0 \quad V(r) = -1650.6\frac{e^{-4x}}{x} + 6484.2\frac{e^{-7x}}{x}$$

$$l = \text{odd} \quad V(r) = -933.48\frac{e^{-4x}}{x} + 4151.1\frac{e^{-6x}}{x} \qquad 2.5$$

$$l = \text{even}, \neq 0 \quad V(r) = -12.322\frac{e^{-2x}}{x} - 1112.6\frac{e^{-4x}}{x} + 6484.2\frac{e^{-7x}}{x}.$$

To check the validity of this assumption we have used equation 2.5 ($l = 0$) to construct the ΛN cross section for center of mass energies up to 20 MeV and the

results are shown in Figure 3. The curve A is obtained by using equation 2.5, whereas curve B is obtained by employing a meson theoretical potential. The potential 2.5 gives a rather poor fit to the experimental data.

As far as the $l \neq 0$ waves are concerned, one cannot make the same statement or for that matter any statement because we do not possess the high-energy data to determine the strength of the P and D waves. However, if these potentials are compared with the ones obtained by exchange of heavy mesons after using a phenomenological hadron-meson Lagrangian, the result is found that the former potentials are on the average too repulsive.

In case C, it was recognized that the ΛN interaction (as well as the $\Lambda\Lambda$) cannot be as strong as the NN case, because the ΛN system does not form a bound state (Brown et al 1970). To this end, an arbitrary factor of $\frac{9}{10}$ was introduced to decrease the attraction when dealing with hyperons. As pointed out by the author himself, this reduction of 10% has no theoretical basis: it is simply a change of an order of magnitude to study the sensitivity of the concentrations.

Even though a multiplicative factor is quite arbitrary, a more justifiable number would actually be either $\frac{2}{3}$ or $\frac{4}{9}$ as explained below. Model C is at any rate more credible than either A or B.

It is important to notice that the energy per particle has a rather strange behavior in models A and B, because it decreases with density, indicating a negative pressure. Such a strange behavior is not present in model C, which, as we said, is slightly more realistic. The sensitivity of the concentrations upon the form of the potential was checked in another paper by Pandharipande & Garde (1972), who introduced tensor forces into the problem. The resulting composition is shown in Figure 4.

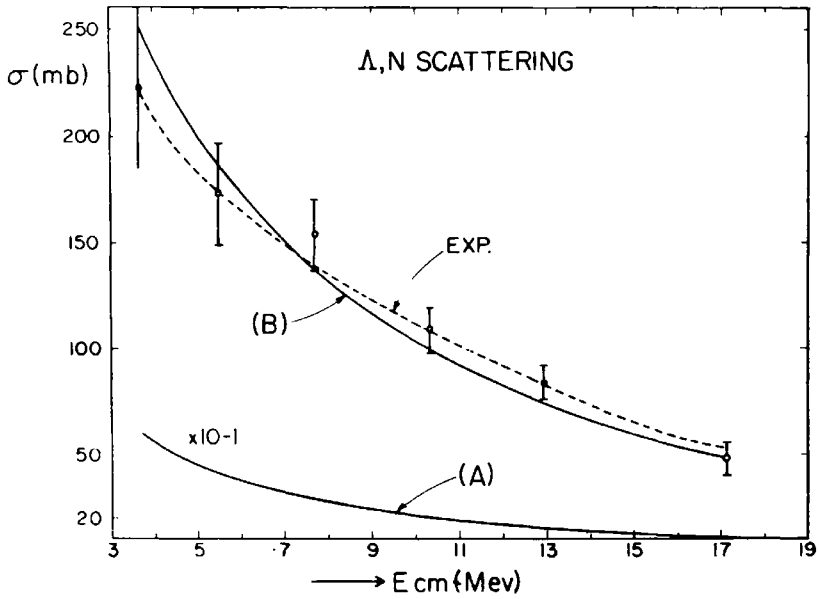


Figure 3 The ΛN scattering cross section as computed by the author (unpublished). Curve A is obtained using $V(r)$ as given by 2.5. The fit is rather poor. Curve B is obtained by using Canuto & Datta (1974) hyperonic potential.

The energy per baryon in C is slightly smaller than in the pure neutron case, but the pressure vs energy-density relation is not sensibly altered. This is the first indication that the hyperonic liquid might not have a great effect in determining the mass and the radius, even though it can be important for the dynamics of the star. As an example, we can quote the work of Langer & Cameron (1969), who showed that the vibrational energy of neutron stars with hyperons is damped very rapidly (on an astronomical scale). For example, a neutron star with $\rho_c > 2.6 \times 10^{14}$ g cm^{-3} will not sustain vibrations.

2.4.5 THE WORK OF MOSZKOWSKI (1974) The problem of self-consistency was not solved in this paper either, but at least its importance was fully appreciated and several physical hypotheses were introduced to account for its presence. It was explicitly recognized that the potential felt by any particle should depend not only upon a second particle, but also upon a third one and so on and even more so at high density, where the Pauli principle loses its importance and the short-range repulsion becomes of primary importance through the dispersion effect. A many-

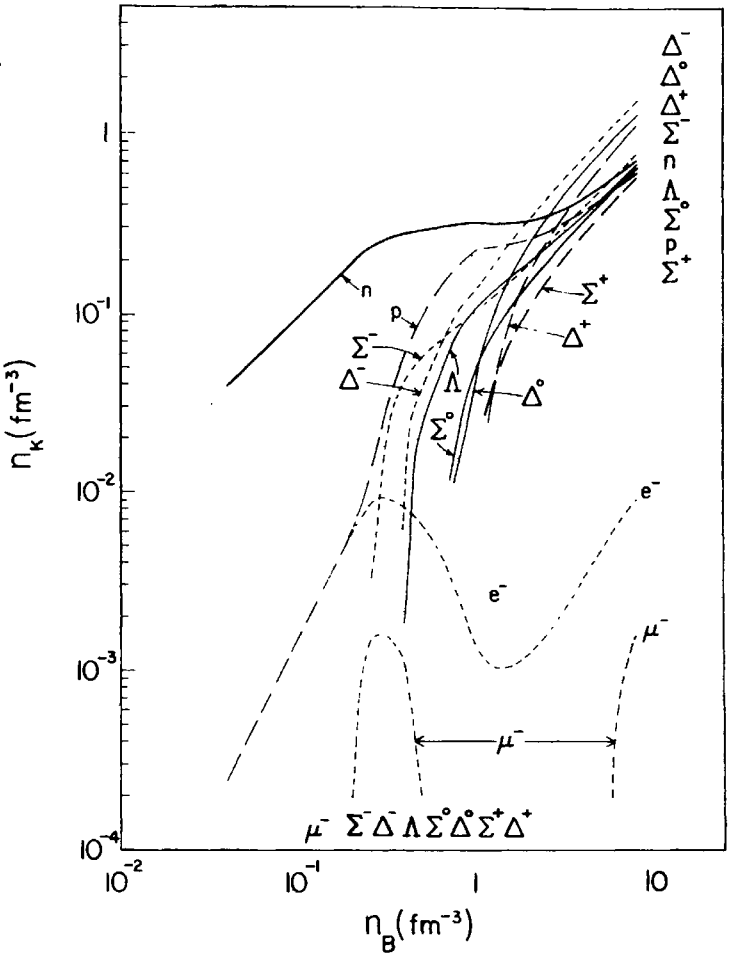


Figure 4 Hyperonic concentrations vs baryonic density as from Pandharipande & Garde (1972).

body treatment of the type explained before was adopted, but the problem of full self-consistency was avoided by choosing a specific form for the G-matrix between any two baryons. This was done in the following way: when the system is composed of only neutrons and protons, the G_{ik}'s ($i, j = n, p$) are taken to be dependent upon n_n+n_p. However, when hyperons are present, the extension is not trivial and the author pursued the calculations using two different parameterizations for the density dependence of the G-matrix. In the first one, he chose (a)

$$G_{pp}(n, p, \Sigma, \text{etc}) \equiv G_{pp}[n+p, p/(n+p)]$$
$$G_{\Sigma\Sigma}(n, p, \Sigma, \text{etc}) \equiv G_{pp}[n+\Sigma, \Sigma/(n+\Sigma)]$$

for identical baryons, and

$$G_{ij} \equiv G_{np}\left[n_i+n_j;\ \frac{\min(n_i+n_j)}{n_i+n_j}\right]$$

for unlike particles.

A second parameterization was chosen by virtue of which the G-matrix depends upon the total baryonic number, that is, (b)

$$G_{nn}(n, p, \Sigma, \text{etc}) \equiv G_{nn}(n_B, n/n_B)$$
$$G_{pp}(n, p, \Sigma, \text{etc}) \equiv G_{pp}(n_B, n_p/n_B)$$
$$G_{\Sigma\Sigma}(n, p, \Sigma, \text{etc}) \equiv G_{pp}(n_B, n_\Sigma/n_B)$$

for identical particles, and

$$G_{ni} \equiv G_{np}(n_B, n_i/n_B)$$
$$G_{ij} \equiv G_{np}(n_B, \{n_i+n_j\}/n_B) \qquad (i, j \neq n)$$

for unlike particles.

Choices a and b reduce the knowledge of any $G_{\alpha\beta}$ to the knowledge of G_{nn}, G_{pp}, and G_{np}, the neutron-neutron, proton-proton, and neutron-proton G-matrices. These were taken from unpublished work of T. Sawada and C. W. Wong, who extended their original nuclear-matter program to the case of unequal numbers of neutrons and protons.

Four tables are given by the author for the quantities G_{nn}, G_{pp}, $G_{np}(T = 0)$, and $G_{np}(T = 1)$, for four values of y defined as

$$y(n_n+n_p) = n_p \qquad y = 0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}$$

and six values of Q_f defined as

$$2Q_f^3 = 3\pi^2(n_n+n_p) \qquad Q = 0.5, 1, 1.5, 2, 2.5, 3.$$

In the Sawada-Wong extension of the $n \neq p$ case, the self-consistency was not fully accounted for in that the single-particle energies entering in the Green's functions were parameterized in a form dependent upon three coefficients fitted to the nuclear matter case ($y = \frac{1}{2}$) and pure neutron case ($y = 0$).

As far as one can see, the self-consistency problem was treated, if not exactly, at least extensively enough to give one the feeling that the sensitivity of the

problem to this aspect has been brought to light. Evidently a final answer can be obtained only after a fully self-consistent approach has been made.

The second serious difficulty encountered by Moszkowski concerns the potentials involving hyperons. Here too, the author decided that, lacking any fundamental approach to the problem, the most sensible way to proceed was to test the results under different assumptions.

The most obvious one is to suppose that the interaction between any two like (or unlike) baryons is the same as the one between $p-p$ (or $n-p$). This will be referred to as choice 1. We have already discussed the poor fit given by such a hypothesis to the ΛN cross section. It suffices to mention that such a postulate (combined with either choice a or b of the G-matrix) gave rather strange results, for instance, negative energy or negative pressure.

These results are a clear indication of something that we already knew: hyperonic forces are essentially different from nucleon-nucleon forces. Faced with this problem, Moszkowski altered the Reid potential in a clever way to incorporate the different nature of the mechanism of exchange of mesons occurring when we deal with hyperons (choice 2). It is known that the intermediate part of the NN potential is dominated by the exchange of a two-pion system. Traditionally this effect is simulated by the exchange of a σ-meson. According to the quark model a π^0 is made of

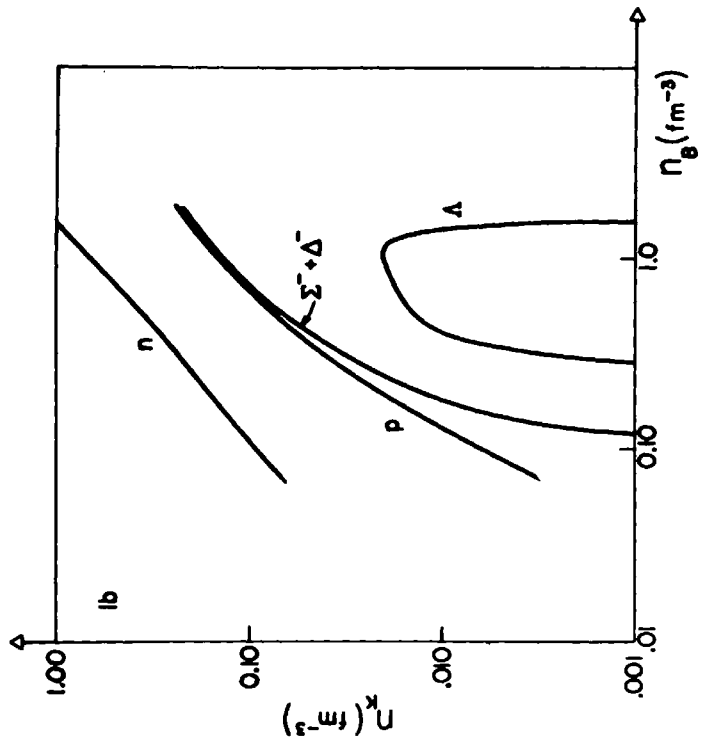
$$2^{-1/2}(n\bar{n}-p\bar{p}),$$

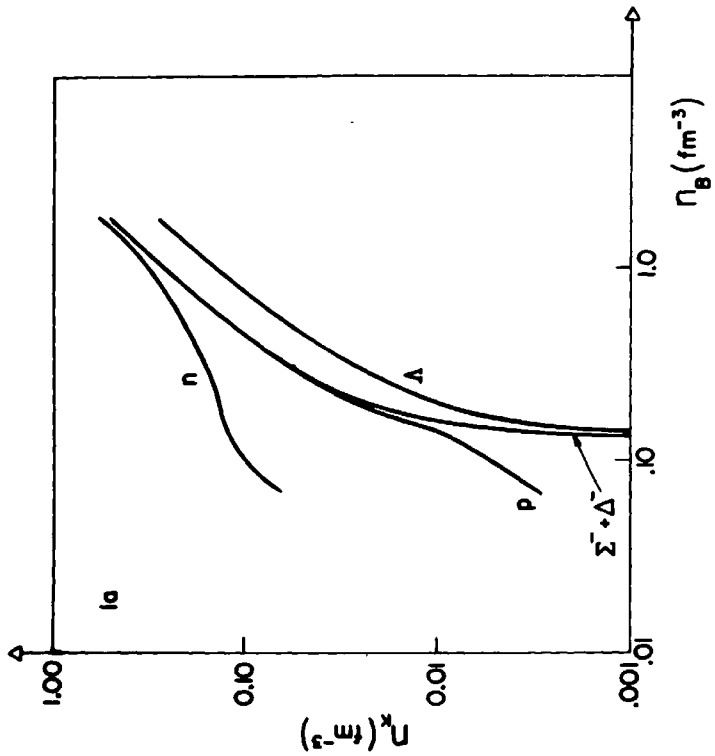
where n is a neutron quark, and $\bar{n}$ is an antineutron quark. Given a vertex containing the same hyperon (in and out), the coupling constant for Λ and Σ is then $\frac{2}{3}$

Table 1 Models 2*a* and *b* of Moszkowski (1974)[a]

n_B	n	p	$\Sigma^-+\Delta^-$	Λ	E/A (MeV)
Model 2*a*					
0.068	0.065	0.003	0	0	7.67
0.131	0.124	0.008	0	0	11.6
0.228	0.164	0.035	0.029	0	13.4
0.362	0.219	0.074	0.069	0	14.0
0.541	0.279	0.132	0.129	0	22.4
0.770	0.365	0.203	0.202	0	48.2
1.056	0.450	0.303	0.303	0	99.7
1.825	0.757	0.534	0.534	0	335
Model 2*b*					
0.068	0.065	0.003	0	0	7.67
0.228	0.196	0.021	0.010	0	17.4
0.541	0.404	0.075	0.062	0	50.1
1.056	0.824	0.131	0.101	0	174
1.824	1.625	0.166	0.033	0	457

[a] The baryonic density and concentrations are in particles fm^{-3}.





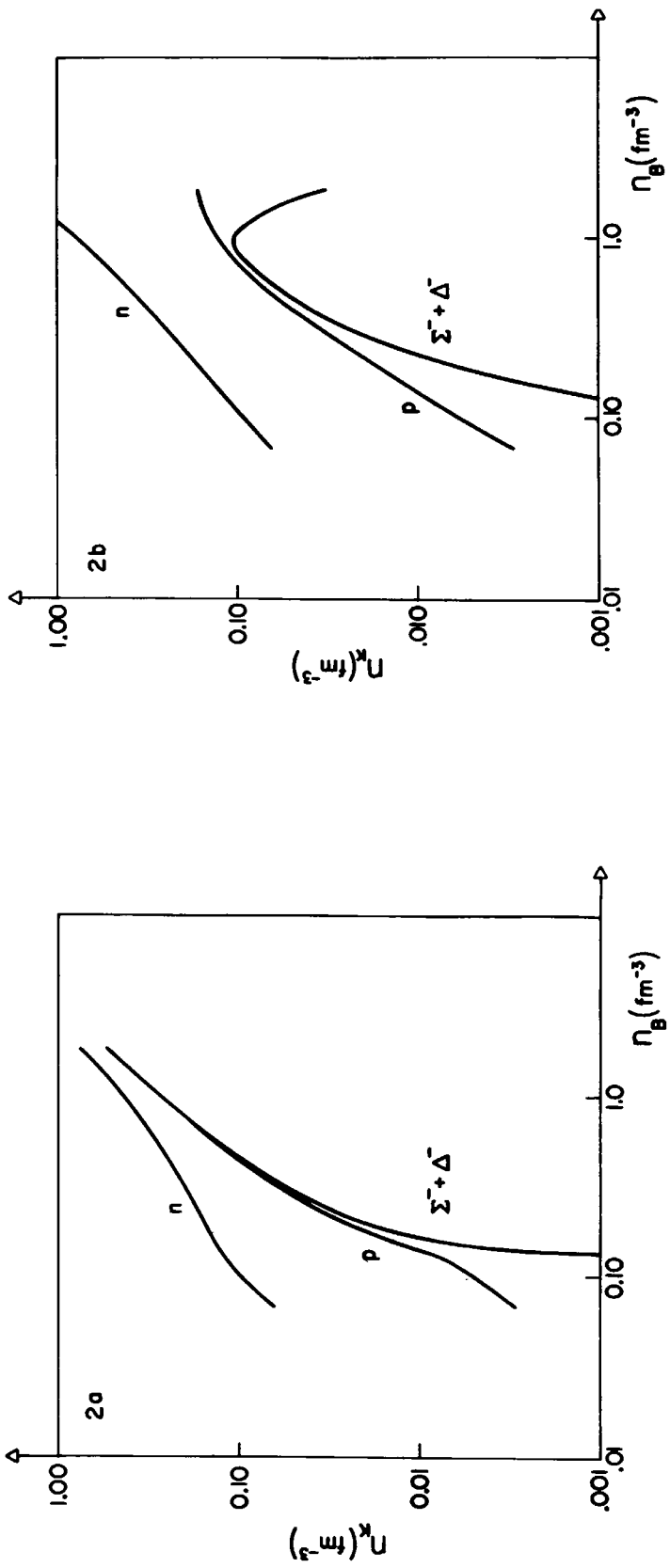


Figure 5 Hyperonic concentrations vs baryonic density following the models 1*a,b* and 2*a,b* of Moszkowski (1974).

as large with respect to N or Δ because Λ and Σ contain two, whereas N and Δ contain three, nonstrange quarks. Thus the reaction matrix for ΛP scattering is $\frac{2}{3}$ of NP and the one for $\Lambda\Lambda$ is $\frac{4}{9}$ of the one for NP. This factor is sensibly different from $\frac{9}{10}$ employed before. As an example, Reid's $^1S_0$ potential has a depth of $\sim$90 MeV, whereas for ΛN derived from meson exchange it is only $\sim$60 MeV, a factor of about $\frac{2}{3}$. The numerical coincidence indicates that this line of reasoning contains some truth.

The resulting energy per baryon and relative concentration of each species vs the baryonic number are presented in Table 1, for the choices 2*a* and *b*. For the reasons given before, the choices 1*a* and *b* seem less credible than the others. Between 2*a* and *b* there is a difference in the density dependence of the *G*-matrix and at the time of this writing it is difficult to decide with certainty in favor of any of them. The relative concentrations are presented in Figure 5.

2.4.6 THE WORK OF BETHE & JOHNSON (1974) As we have repeatedly stressed, the most important ingredient in any many-body computation is the NN potential. This is no place to discuss this topic; it suffices to say that among the most simple forms so far proposed, the one published by Reid has been most successful in fitting the experimental phase shifts. Reid's potential is a superposition of Yukawas of different strengths and ranges, and even though the analytic structure is highly suggestive of those derived theoretically from meson theories, still its purely phenomenological nature is evident in the fact that one cannot write it in the general form

$$V_{LSJ}(r) = V_c(r) + V_\sigma \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + V_T S_{12} + V_{LS} \mathbf{L}\cdot\mathbf{S} + V_Q(\mathbf{L}\cdot\mathbf{S})^2, \qquad 2.6$$

but one must be content with apparently unrelated forms for each partial wave. On the basis of the meson theory of nuclear forces, the ω-vector meson should dominate the behavior of $V_{LSJ}(r)$ at short distances. Moreover, because ω is (vector) isoscalar, the core should be the same in all states, the range of it should correspond to that of the ω's, and finally the strength of it should be given by the ω-coupling constant. None of these features is present in Reid's potential. Because at high densities the behavior of the core is of critical importance, Bethe and Johnson undertook the task of rebuilding an NN potential that would satisfy the previous requirements. In Table 1 of their paper, five sets of coefficients C_n and n are given, entering in the definition of the Bethe-Johnson potential ($x = 0.7r$):

$$V_{LSJ}(r) = \sum_n C_n(LSJ)\frac{e^{-nx}}{x} + V_T(r). \qquad 2.7$$

The coefficient C_1 and the tensor V_T are taken from the one-pion-exchange model.

The new potentials reproduce experimental phase shifts, nuclear matter binding energy, and deuteron quadrupole moment as well as Reid's. The work of Bethe and Johnson consists of two distinct parts: 1. determination of the ground-state energy and therefore equation of state for a pure neutron gas; and 2. derivation of the same properties when hyperons are present. Let us first discuss the results for a pure neutron liquid.

The many-body technique employed is the lowest-order constraint variation (LOCV) as developed by Pandharipande (1971). It is stated by the authors that such a method is a good approximation of the more exact treatment developed by Pandharipande & Bethe (1973) and, moreover, that LOCV is entirely satisfactory for numerical calculations. We shall see, however, how recent computations have shown that this is indeed not the case and that such a method gives too large an energy when compared with an essentially exact Monte Carlo computation. Let us first present the results and then discuss this point.

Five different models have been studied and we cannot but give the summary of their results. For Model I an average process was used, whereby only even and odd potentials were used for any two baryons. The resulting equation of state reads (n_B in fm^{-3}, E_B in MeV, P_{35} in units of 10^{35} dynes cm^{-2}; $x = n_B$, $a = 1.54$)

$$E_B = 236x^a, \qquad P_{35} = 5.831x^{a+1}, \qquad c_s^2 = x^a(1.01+0.648x^a)^{-1}. \tag{2.8}$$

When compared with the values for a pure neutron gas, given in Table 10 (Part 1) obtained by using Reid's potential, equation 2.8 gives values consistently larger. For example, at $n_B = 1$, we read from Table 10 (Part 1) that $E_B = 157.47$ and $P_{35} = 4.029$, whereas Model I predicts $E_B = 236$ and $P_{35} = 5.831$. The present equation of state is much stiffer. Moreover, the velocity of sound c_s exceeds the velocity of light at $n_B = 1.98$ fm^{-3}. In Model II, the average over the potentials was not performed and a more correct evaluation was carried out of the ^3P_J waves. The resulting energies were much lower than in Model I, but still higher than those of Table 10 (Part 1).

Model III yields results very similar to Model II, the main difference being the treatment of the $^1P_1$ wave, that is here less repulsive. Again, E_B is higher than that given in Table 10 (Part 1).

Models IV and V are interesting in that the range of the repulsion corresponds exactly to $m_\omega c/\hbar = 3.85$. Model V has a less repulsive core in even states and a more repulsive core in odd states with respect to Model IV. The resulting energies for particles do not differ significantly as shown in Table 2, where the results of all the models are collected. The same many-body technique was employed by the authors to study a hyperonic liquid composed of p, Λ, Σ, and Δ. The hyperonic

Table 2 Energy per particle (MeV) for neutron (N) and hyperon (Y) liquids following different models of H. A. Bethe and M. B. Johnson (1974)

	II		III		IV		V		Reid
n_B (fm^{-3})	N	Y	N	Y	N	Y	N	Y	N
0.5	65	65	63	63	61	61	63.5	63.5	57
1.0	189	189	185	189	181	191	191	185	156
2.0	610	690	590	560	560	575	570	480	490
4.0	1890	1920	1810	1610	1580	1450	1650	1200	1450
10.0	7500	7300	7200	6200	6600	5900	6000	4000	—

potential was constructed in the following way: the authors define an even- and an odd-state potential such that the even one is the same as the $^1D_2$ potential for the NN case and the odd one is a spin-isospin average of the ^3P_J states NN potential (less OPEP).

The authors do not give a numerical comparison of the results that such potentials would yield for experimental hyperonic data such as the one presented in Figure 3 for the ΛN cross section and we are therefore in no position to judge the reliability of such a hyperonic potential. Granting, without justification, that one can use the NN potential in some way, the even state has been chosen here more correctly than in Pandharipande (1971), because $^1D_2$ is less attractive than $^1S_0$ and it therefore resembles more closely the true ΛN potential as derived from meson exchange.

We feel less confident about the procedure used for the odd waves. In fact, it is known that $^3P_1$ for NN is repulsive at any distance, whereas the $^3P_1$ theoretically

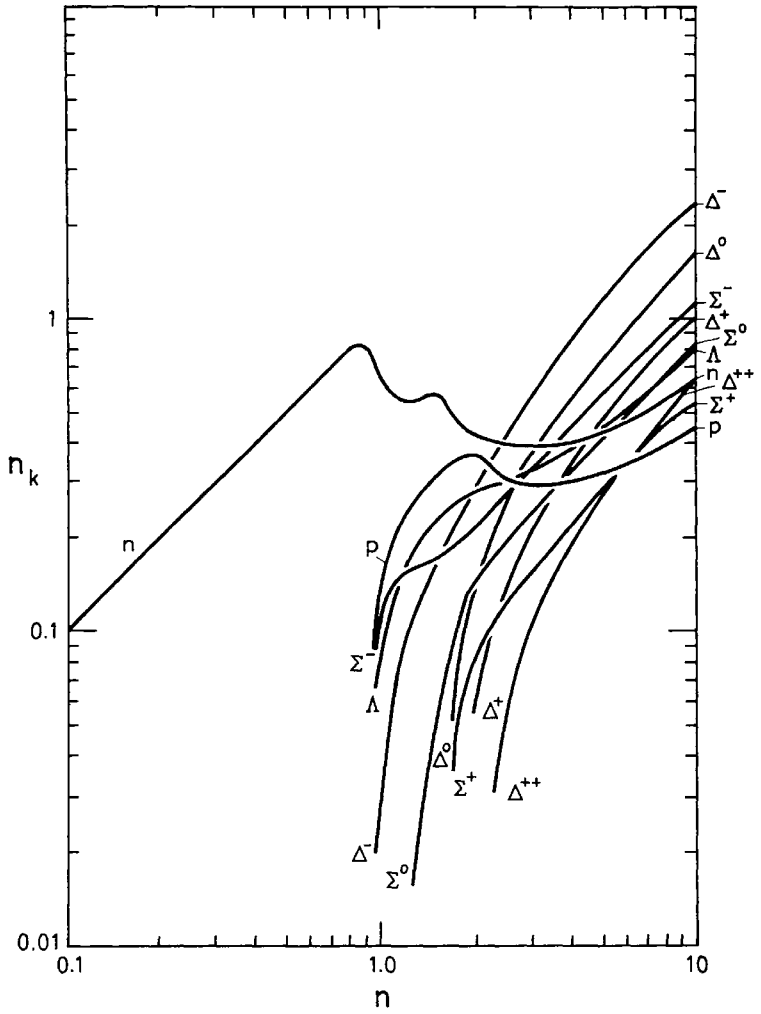


Figure 6 Hyperonic concentrations (in particles fm^{-3}) vs baryonic density (same units) as from the work of Bethe & Johnson (1974).

derived (OBEP) for ΛN is attractive and repulsive. It therefore seems that the odd potential used by Bethe and Johnson is too repulsive. The result for the ground-state energy when hyperons are included is also given in Table 2 and the relative concentrations (for Model V) are presented in Figure 6.

Before turning to the equation of state, we ought to comment on the many-body theory employed. A hypothetical but indicative calculation for a pure neutron liquid with a simplified potential of the form ($x = 0.7r$),

$$V(x) = 6484.2 \frac{e^{-7x}}{x}, \qquad 2.9$$

has been recently solved by several groups, with the aim of revealing the approximations used in the different many-body techniques so far adopted. If we take the Monte Carlo results of Cochran & Chester (1973) as a gauge to judge the other methods, the results are as follows (Figure 7):

1. The energies obtained by using LOCV of Pandharipande (1973) are higher than the Monte Carlo energies up to 200 MeV at 3.2×10^{15} g cm^{-3} (curve 1). The LOCV method has been employed by Bethe & Johnson (1974)
2. The many-body technique employed by Shen & Woo (1974) yields results that never exceed the Monte Carlo energies by more than 25 MeV (curve 2)
3. The full variational method (and not just the LOCV) as developed by

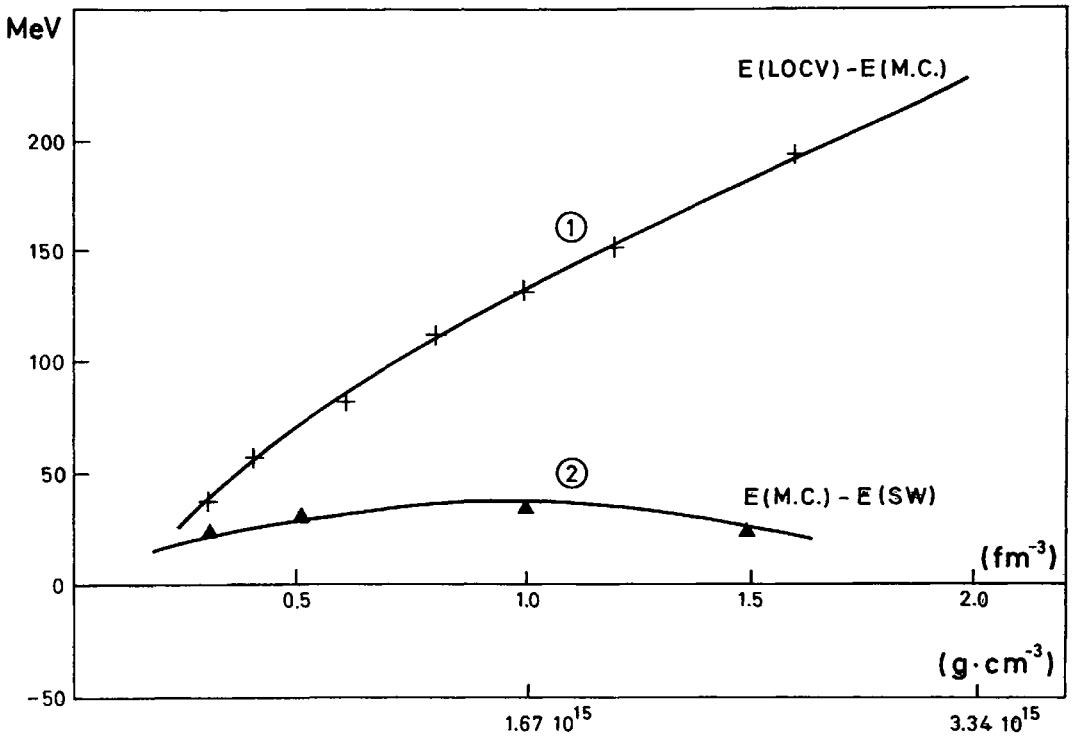


Figure 7 Energy per baryon of a neutron liquid as from Pandharipande (1973) (LOCV) and Shen & Woo (1974). Both results are judged against the almost exact Monte Carlo (MC) results of Cochran & Chester (1973). The LOCV yields too high energy.

Pandharipande & Bethe (1973) gives energies in agreement with Monte Carlo results (H. A. Bethe 1974, personal communication)

In view of the much higher reliability of the methods employed in 2 and 3, it would be highly desirable to repeat the Bethe & Johnson computations using the new potentials but employing the more exact many-body formulation either in the form 2 or 3. An overall concern about the Bethe-Johnson potential is that the ω-meson coupling constant $g_\omega^2/4\pi\hbar c$ is 22.1, 29.6, 47, 47, and 47 for models V, IV, III, II, and I, respectively. This has to be compared with the "experimental value" of 10 ± 2.

2.5 *Concluding Remarks on the Hyperonic Liquid*

Despite the considerable amount of work that has gone into the study of the hyperonic liquid, none of the previous works can claim the same degree of reliability

Table 3 ΛN potential as derived by meson exchanges[a]

	$\eta(548, T=0, Ps)$	$k(495, T=1/2, Ps)$
	$\eta \equiv 2.78\, f^{-1}$	$k \equiv 2.35\, f^{-1}$
V_σ	$-19.39(e^{-\eta r}/r)$	$+(-)^{L+S}57.7(e^{-kr}/r)$
V_T	$-58.18 f(r)(e^{-\eta r}/r)$	$+(-)^{L+S}173.3 f(r)(e^{-kr}/r)$
	$f(r) = \frac{1}{3} + 0.36/r + 0.13/r^2$	$f(r) = \frac{1}{3} + 0.43/r + 0.18/r^2$
	$V_c = V_{LS} = V_Q = 0$	
	$\sigma(490, T=0, S)$	$\omega(888, T=0, V)$
	$\sigma \equiv 2.48\, f^{-1}$	$\omega \equiv 4.51\, f^{-1}$
V_c	$1131.08(e^{-\sigma r}/r)$	$+6815.03(e^{-\omega r}/r)$
V_σ	$+0.98 f(r)(e^{-\sigma r}/r)$	$+[616.78 - 43.61 f(r)](e^{-\omega r}/r)$
	$f(r) = 0.402/r + 0.32/r^2 + 0.13/r^3$	$f(r) = 0.22/r + 0.09/r^2 + 0.02/r^3$
V_T	0	$-925.18 f(r)(e^{-\omega r}/r)$
		$f(r) = \frac{1}{3} + 0.22/r + 0.05/r^2$
V_{LS}	$-135.90 f(r)(e^{-\sigma r}/r)$	$-5754.49 f(r)(e^{-\omega r}/r)$
	$f(r) = 0.40/r + 0.16/r^2$	$f(r) = 0.22/r + 0.05/r^2$
V_Q	$-1.98 f(r)(e^{-\sigma r}/r)$	$+87.23 f(r)(e^{-\omega r}/r)$
	$f(r) = 0.16/r^2 + 0.13/r^3$	$f(r) = 0.05/r^2 + 0.02/r^3$
	$k^*(890, T=1/2, V)$	
	$k^* \equiv 4.42\, f^{-1}$	
V_c	$(-)^{L+S}552.41(e^{-k^* r}/r)$	
V_σ	$(-)^{L+S}[353.10 - 147.82 f(r)](e^{-k^* r}/r)$	$f(r) = 0.22/r + 0.10/r^2 + 0.02/r^3$
V_T	$-(-)^{L+S}529.65 f(r)(e^{-k^* r}/r)$	$f(r) = \frac{1}{3} + 0.22/r + 0.05/r^2$
V_{LS}	$-(-)^{L+S}1495.69 f(r)(e^{-k^* r}/r)$	$f(r) = 0.22/r + 0.05/r^2$
V_Q	$(-)^{L+S}295.65 f(r)(e^{-k^* r}/r)$	$f(r) = 0.05/r^2 + 0.02/r^3$

[a] Λ, N potential: $T = 1/2$; $V = V_c + V_\sigma \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + V_T S_{12} + V_{LS} \mathbf{L} \cdot \mathbf{S} + V_Q W_{12}$.

that characterizes that of the pure neutron liquid. One of the main reasons is the inadequate treatment of the hyperonic forces. A full OBEP hyperonic potential has recently been determined by Canuto & Datta (1974) on the basis of data from Brown, Downs & Iddings (1970). The general potential is written in the form 2.6 and the contributions from the several bosons are listed separately. As an example, we present in Table 3 the ΛN potential that was used in the construction of curve *B*, Figure 3. The equilibrium calculations with such a potential, however, have not yet been performed. As far as the many-body techniques are concerned, Figure 7 clearly indicates that the LOCV should be replaced with either method 2 or 3 just mentioned.

A final criticism of all the previous computations has been raised by Sawyer (1972). He has stressed the point that in deciding whether a given hyperon will appear or not, a parameter equally as important as the potential, and perhaps even more, is the mass shift that every hyperon acquires because of the dense surrounding medium. In a model calculation for Δ^-, Sawyer has shown that the mass shift is of such a magnitude as to quench the appearance of Δ^- until densities higher than 10^{16} g cm^{-3} (see Figures 2 and 4). None of the previous computations has included any mass shift and only when such an effect is thoroughly investigated and embodied in the calculations can the previous results be totally trustworthy.

2.6 *The Equation of State with Hyperons*

We have already presented in Tables 11 and 12 of Part 1 two $P = P(\varepsilon)$ relations including hyperons. To these we should now add the one computed by Bethe and Johnson. This is given in Tables 4*a* and *b*.

Moszkowski's results [Table 12 (Part 1)] can be compared with either Tables 11 (Part 1) or 4 only numerically, because both the many-body theory and the baryonic potentials are different. We have already commented previously about the many-body theory employed in Tables 11 (Part 1) and 4. The same type of comment cannot be made concerning Moszkowski's theory. However, we feel that as far as the hyperonic potential is concerned, his work is more realistic.

In view of the difficulties still plaguing all the hyperonic equations of state published so far, it is our feeling that it is safer not to use any of them as yet and consider the region $>2 \times 10^{14}$ g cm^{-3} as a pure neutron liquid, thus using Tables 10 (Part 1) or 4*a* of this paper up to a point where the relativistic effects become important (see Section 4).

3 THE SOLID CORE

3.1 *Generalities*

The uncertainties in the results presented in the previous section, though important, are of a rather technical nature, regarding both the similarity (or lack of similarity) of the full hyperonic potential to the *NN* case and the so far little investigated mass-shift effect. In both cases the physical idea is clear. Time and an extra good deal of work will almost certainly clarify the situation. On the other hand, the region we are about to discuss has been characterized by a much deeper, more fundamentally

Table 4a Equation of state for a pure neutron gas after Bethe & Johnson (1974)

n_B (fm^{-3})	ρ ($g\,cm^{-3}$)	$\mathscr{E}$ (MeV)	P (dynes cm^{-2})	$P/\rho c^2$
0.1	1.70×10^{14}	12.6	1.19×10^{33}	—
0.15	2.55×10^{14}	16.6	2.93×10^{33}	0.013
0.2	3.42×10^{14}	21.2	6.00×10^{33}	0.019
0.25	4.31×10^{14}	26.0	1.09×10^{34}	0.028
0.3	5.20×10^{14}	32.2	1.83×10^{34}	0.039
0.4	7.04×10^{14}	46.9	4.09×10^{34}	0.087
0.5	8.95×10^{14}	64.4	7.61×10^{34}	0.095
0.6	1.09×10^{15}	83.7	1.26×10^{35}	0.128
0.7	1.31×10^{15}	109.0	1.99×10^{35}	0.169
0.8	1.53×10^{15}	135.0	2.85×10^{35}	0.207
0.9	1.76×10^{15}	160.0	3.71×10^{35}	0.234
1.0	2.01×10^{15}	190.0	4.92×10^{35}	0.272
1.1	2.28×10^{15}	224.0	6.23×10^{35}	0.304
1.25	2.70×10^{15}	274.0	8.58×10^{35}	0.353
1.4	3.16×10^{15}	327.0	1.14×10^{36}	0.401
1.5	3.48×10^{15}	360.0	1.34×10^{36}	0.428
1.7	4.19×10^{15}	442.0	1.85×10^{36}	0.493
2.0	5.35×10^{15}	560.0	2.76×10^{36}	0.574
2.5	7.70×10^{15}	788.0	4.83×10^{36}	0.698
3.0	1.06×10^{16}	1040.0	7.62×10^{36}	0.800

physical question. Do neutrons solidify at high enough density? We have so far treated the neutrons and for that matter the hyperons as a liquid even though it is perfectly legitimate to ask oneself whether the NN potential is strong enough to force nucleons into a regularly arranged structure.

The old hard-core potential almost by definition gives rise to a solid structure at densities of the order of $n_B = (\frac{4}{3}\pi r_c^3)^{-1}$. In fact, because at these densities each particle feels an infinitely strong repulsion everywhere around it, it is caged, and such a localization leads to a crystalline structure. Such a potential is therefore of no interest, in addition to having been superseded by more modern versions that call for the existence of a soft core due to the exchange of vector mesons. With a soft but infinite potential, like the ones constructed by Reid and by Bethe and Johnson, it is not clear if the system will eventually crystallize, because the potential can be too soft, can let the particles slip through it, and can never produce the localization necessary for a crystal structure to set in, unless perhaps when the distance is exceedingly small and the repulsion exceedingly high. If that is the case, it could well be that a crystallization density exists but is so high that it is of no interest even for the superdense interior of neutron stars. A third possibility is that the repulsive potential is finite at the origin, that is, a gaussian type of potential of the form proposed by the Japanese school (Otsuki et al 1964). If the gaussian form is such that it provides enough repulsion to localize the particles, it will

Table 4b Equation of state for hyperonic matter after Bethe & Johnson (1974)

n_B (fm^{-3})	ρ (g cm^{-3})	$\mathscr{E}$ (MeV)	P (dynes cm^{-2})
0.1	1.70×10^{14}	12.6	1.19×10^{33}
0.15	2.55×10^{14}	16.6	2.93×10^{33}
0.2	3.42×10^{14}	21.2	6.00×10^{33}
0.25	4.31×10^{14}	26.0	1.09×10^{34}
0.3	5.20×10^{14}	32.2	1.83×10^{34}
0.4	7.04×10^{14}	46.9	4.09×10^{34}
0.5	8.95×10^{14}	64.4	7.61×10^{34}
0.6	1.09×10^{15}	83.7	1.26×10^{35}
0.7	1.31×10^{15}	109	1.99×10^{35}
0.8	1.53×10^{15}	134	2.84×10^{35}
0.9	1.76×10^{15}	160	3.36×10^{35}
1.0	2.01×10^{15}	189	4.02×10^{35}
1.1	2.26×10^{15}	215	5.02×10^{35}
1.25	2.66×10^{15}	254	6.76×10^{35}
1.4	3.08×10^{15}	296	8.81×10^{35}
1.5	3.38×10^{15}	324	1.03×10^{36}
1.7	4.01×10^{15}	382	1.38×10^{36}
2.0	5.04×10^{15}	475	2.02×10^{36}
2.5	7.04×10^{15}	640	3.40×10^{36}
3.0	9.38×10^{15}	815	5.20×10^{36}

cease to do so when the energy of localization, $\varepsilon = \hbar c n_B^{1/3}$ is greater than V_0, the value of the potential at $r = 0$. For $V_0 = 1$ GeV, $n_B \approx 10^3$ fm^{-3}, or $\rho \approx 10^{18}$ g cm^{-3}. It is not clear a priori whether a repulsive gaussian will produce a crystal, but if it does it will be for only a finite interval in density. For densities such that $\varepsilon \gg V_0$, the crystalline structure ceases to exist.

Because the mesonic theories of nuclear forces indicate that, near the origin, the potential provided by the ω-meson has a Yukawa shape, in what follows we consider exclusively such types of potentials.

The work on the solidification problem can be divided into two categories: the first is of an exploratory nature, whereas the second is of a microscopic nature, because it employs the best available nucleon-nucleon data and many-body techniques. Before reviewing the work, we should perhaps comment on the results: all but one of the computations so far performed indicate a solidification density well within the range of the values expected in the interior of neutron stars, the lowest being 5×10^{14} g cm^{-3} and the highest 3×10^{15} g cm^{-3}. We first review the work in some detail and then critically summarize the results in Section 3.3.

3.2 *Review of the Results*

3.2.1 THE WORK OF CAZZOLA, LUCARONI & SCARINCI (1966) As far as this author can recollect, this work represents the first published paper in which the idea of a

solid structure was proposed and a model computation of the corresponding equation of state was carried out. The authors do not actually show that a crystal structure indeed occurs. They postulate that the nucleon-nucleon potential is repulsive enough to produce a localization, which is then accounted for by assuming that each nucleon is trapped in a finite region of space, characterized by a potential $U(r)$. The single-particle energies are found by solving a one-particle Dirac equation in which $U(r)$ is taken to be a square well. Once the eigenvalues are known, the pressure is easily evaluated with the result

$$P = 0.32 n_B^2 10^{39} \text{ dynes cm}^{-2}, \qquad 3.1$$

where n_B is the baryonic number density (particles per fm^3). This expression does not join in any way to the values given in Tables 10–12 (Part 1), giving $P = 6.27\ 10^{38}$ dynes cm^{-2} at $n_B = 1.4$, as compared with $1.189\ 10^{36}$ from Table 12 (Part 1). The merit of the present work is clearly not in the expression for the equation of state, but in its having pointed out a physically attractive alternative to the liquid structure.

3.2.2 THE WORK OF BANERJEE, CHITRE & GARDE (1970) These authors performed an exploratory computation of the equation of state for matter at high densities by localizing the neutrons in a bcc lattice and performing a classical lattice-dynamics analysis. The interaction potential between two opposite neutrons was taken to be an average of Reid's $^1S_0$ potential and the repulsive part of the same wave. The total energy is contributed by the rest energy, the static energy, and the vibrational energy. This last contribution required an elaborate lattice-dynamics analysis.

In the spirit of classical lattice dynamics, the solid structure was thought to set in when the oscillator frequency of each nucleon at its lattice site ω, proportional to the second derivative of the NN potential, becomes real. This occurred at 8×10^{14} g cm^{-3}, which was taken to be the solidification density.

If compared with the presently known E/N for the liquid state, the solid energies so obtained are far too large. The comparison is not meaningful, however, since the solid has been computed with zero-spread single-particle wave functions. A fully quantum-mechanical computation was shortly thereafter initiated by Canuto & Chitre (1973a,b), but before its completion, several other results appeared and we review them first.

3.2.3 THE LAW OF CORRESPONDING STATES The approach to the problem of solidification of neutron matter through the application of the so-called law of corresponding states was first suggested by Anderson & Palmer (1971) (see also Palmer & Anderson 1974). It is known that the potential between two rare gas atoms can be represented by a 6-12 function of the Lennard-Jones type (LJ)

$$V(r) = 4\varepsilon \left\{ \left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6} \right\}. \qquad 3.2$$

Because only two parameters, an energy (ε) and a length (σ), are necessary to specify a given substance, de Boer (1948) was able to show that the dimensionless quantities

$$V^* = V/N\sigma^3, \qquad kT^* = kT/\varepsilon, \qquad P^* = P\sigma^3/\varepsilon \tag{3.3}$$

for several known rare gases fall in straight lines when plotted against the quantum parameter $\Lambda = h(m\varepsilon\sigma^2)^{-1/2}$, nowadays known as the de Boer parameter. If, in the same way, we were able to find the Λ corresponding to a system of neutrons, then the solidification pressure would be readily found by simply looking at the experimental (universal) curve P_s^* vs Λ. By so doing, Anderson & Palmer (1971) found the following result:

$$P_s = 0.4\varepsilon\sigma^{-3} = 4 \text{ MeV fm}^{-3} = 6.7 \times 10^{33} \text{ dynes cm}^{-2}; \Lambda \approx 3.4. \tag{3.4}$$

The similarity with He^3 ($\Lambda \sim 3.5$) is striking. The authors arrived at the result without fitting an NN potential to a LJ function. This was later attempted by Clark & Chao (1972), who constructed a density-dependent NN potential that is a superposition of $^1S_0$ and $^1D_2$ waves. When such a potential was fitted to a LJ form, the following results were found:

$$P_s = 4.7 \times 10^{33} \text{ dynes cm}^{-2}; \Lambda = 5.3. \tag{3.5}$$

The corresponding solidification densities can be found by continuing the $P = P(\varepsilon)$ relation given in Table 10 (*Part* 1) with the solid structure. The results are

$$\rho_s = 3.7 \times 10^{14} \text{ g cm}^{-3}; \ \rho_s = 3 \times 10^{14} \text{ g cm}^{-3} \tag{3.6}$$

for Anderson and Palmer and for Clark and Chao, respectively. We critically review these results in Section 3.3.

3.2.4 THE WORK OF COLDWELL (1972) In a paper submitted in April 1971, Coldwell performed an exploratory computation with the goal of investigating whether a simple Hartree-Fock computation would predict neutron solidification. The analysis is incomplete with respect to the work described in the following sections, because it neglects the distortion of the two-body wave function caused by the short-range repulsion and accounted for by the introduction of a correlation function of the type described in Section 5.2 of Part 1. An analogous computation was performed years back by Nosanow & Shaw (1962) for ^{3}He. Without a correlation function, the single particle energies are bound to be overestimated by a considerable amount, as happened in solid ^{3}He.

The importance of Coldwell's work lies in the fact that for the first time he performed a double computation, that is, he treated liquid and crystal with the same method, thus making the comparison meaningful. The novelty of this computation with respect to the ones performed before and even after is that the single-particle wave functions were taken to be eigenfunctions of a fictitious potential

$$V(c, x) = c \sin^2(\pi x/a). \tag{3.7}$$

For $c \to 0$, the single-particle wave function becomes a plane wave (liquid), whereas for $c \neq 0$, $V(c, x) \sim x^2$, as $x \to 0$. The corresponding wave functions are then localized gaussians (solid). No analytic expression can be obtained for the ground-state energies (computed using Reid's potential) and the work has to be done entirely numerically. The results can be summarized as follows: Up to a density of

3.98×10^{14} g cm^{-3}, the minimum of the energy is provided by a ferromagnetic arrangement of the nucleons and not by a crystal. For densities higher than 3.98×10^{14} g cm^{-3}, however, the minimum of the energy is given by localizing the particles in a crystalline structure. The energies obtained by Coldwell are too high when compared with the ones given in Table 10 (Part 1) and Table 2. However, the absolute values for the energy of the liquid and solid are probably off by the same amount and consequently the solidification pressure and density can be considered to be correct.

3.2.5 THE WORK OF CANUTO & CHITRE (1973, 1974) The work presented so far has clearly indicated that a full quantum-mechanical computation is needed to study the solidification problem. This is a formidable task in that a microscopic computation can be considered conclusive only if performed simultaneously for liquid and solid with the same many-body theory and the same potential, that is, the ingredients must be identical. There are essentially two many-body techniques presently available. The variational method, extensively used for ^{3}He, has the unfortunate feature of not lending itself to a straightforward handling of spin and angular momentum, features that characterize the nuclear forces. On the other hand, it has the advantage that by imposing certain restrictions on the correlation function, one can handle the higher many-body contributions in such a way as to render their contribution unimportant. There is in fact one computation for ^{3}He (Hetherington et al 1967), where a restricted variational correlation function was chosen to make the three-body clusters reasonably small. Even though this is a brute-force way of doing things, at least one has an empirical way of making the cluster expression converge. We must notice, however, that if the three-body contribution is a serious problem for liquids it is perhaps of less importance for solids.

In systems that are translationally invariant (i.e. liquids) the unperturbed wave functions are usually taken to be plane waves. Brueckner's theory is a low-density expansion, and it is not obvious that such a method should work for solid hydrogen or solid helium. Solid hydrogen has recently been studied by Østgaard (1971, 1972) and the t-matrix approach (including up to two-body clusters) was found to work rather well. This is partly due to the fact that instead of plane waves, one employs gaussian wave functions, which already incorporate several correlations. One can therefore hope that the cluster expansion up to second order will be adequate for solids but not necessarily for liquids.

If the variational and t-matrix techniques are considered up to the second order, the advantages offered by the t-matrix are clearly superior because the state dependence of the NN potentials can be fully accounted for to any degree of accuracy. On the basis of this last feature and the probable unimportance of the three (and more)-body correlations for solids, Canuto and Chitre decided to adopt the t-matrix method. Evidently the whole method can be judged only after having tested it against a well-known quantum solid such as ^{3}He.

It is shown later that such a method actually produces the best E/N vs molar volume so far published, the deviation from the experimental data being less than 1°K.

Even though the author is not aware of any such computation, it is almost certainly true that the same method, when applied to a liquid, say, liquid ^{3}He, would probably produce rather poor results. This poor performance has nothing to do with the situation for the solid, however, where, as we have just said, it works better (or equally well) than any other method. Evidently, the choice of a method that works well for a solid and perhaps fails for a liquid violates the requirement of handling liquid and solid with the same method. Once the energies for the solid are obtained, the only way of determining if such a configuration is actually solid, in the commonly accepted definition, is by studying its elastic properties, that is, finding out if it has resistance against shearing stresses and if so, for which density interval.

The results indicate that the shear modulus C_{44} is positive for densities greater than (1–3) $\times 10^{15}$ g cm^{-3}, thus providing a lower limit for the stability of a neutron crystal.

Let us now consider a system of neutrons described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\sum_i \nabla_i^2 + \tfrac{1}{2}\sum_{i<j} V_{ij}. \tag{3.8}$$

The Slater determinant for the system is built up of single-particle wave functions of the gaussian form

$$\phi(i) = \frac{\alpha^{3/2}}{\pi^{3/4}}\exp\frac{-\alpha^2}{2}|\mathbf{r}_i - \mathbf{R}_i|^2, \qquad \alpha^2 = \frac{m\omega}{\hbar}. \tag{3.9}$$

Here, $\mathbf{R}_i$ is the ith lattice site around which the particle performs an oscillatory motion under the influence of the remaining $(N-1)$ particles. The t-matrix expansion gives the following expression for the energy per particle up to and including two-body clusters:

$$\frac{E}{N} = \tfrac{3}{4}\hbar\omega + \sum_{ij}\frac{\int \psi_{ij}^* V_{ij}\phi_{ij}\, d^3r_i\, d^3r_j}{2N\int \psi_{ij}^*\phi_{ij}\, d^3r_i\, d^3r_j} = \tfrac{3}{4}\hbar\omega + \tfrac{1}{2}\sum_k n_k\varepsilon_k. \tag{3.10}$$

Here ϕ_{ij} is the uncorrelated two-body wave function $[=\phi(i)\ \phi(j)]$, whereas ψ_{ij} is the correlated two-body wave function to be determined by solving the homogeneous Bethe-Goldstone equation ($\boldsymbol{\Delta} = \mathbf{R}_1 - \mathbf{R}_2$)

$$\left[\frac{-\hbar^2}{m}\nabla_r^2 + \tfrac{1}{4}m\omega^2(\mathbf{r}-\boldsymbol{\Delta})^2 + V(r)\right]\psi(\mathbf{r}) = [-\tfrac{3}{2}\hbar\omega - 2U(0)]\psi(\mathbf{r}). \tag{3.11}$$

The most difficult part of the problem lies in the solution of equation 3.11. In fact the term $\mathbf{r}\cdot\boldsymbol{\Delta} = r\Delta\cos\theta$, much like the Stark effect, couples even with odd waves. If an angular momentum expansion is made of $\psi(\mathbf{r})$, then an infinite set of coupled differential equations results.

All previous work that dealt with such an equation invariably averaged over $\mathbf{r}\cdot\boldsymbol{\Delta}$, thus avoiding the angular-momentum problem.

To judge the t-matrix method and the handling of equation 3.11, the most appropriate test is solid ^{3}He. In Figure 8 we present several results. For the time

being the significant comparison should be made between the results of Guyer (1969), who applied the method just described with the $\mathbf{r}\cdot\boldsymbol{\Delta} \approx r\Delta$ approximation, and the results of Canuto et al (1974a), who expanded ψ in its angular-momentum components and solved a resulting set of 25 coupled differential equations. The improvement on the previous result is significant. If the angular-momentum expansion is important for ^{3}He, whose two-body interaction is spherically symmetric, it is even more so when we deal with the NN case, where each partial wave has a different potential.

It is clear that, if we perform an average over the solid-state term $\mathbf{r}\cdot\boldsymbol{\Delta}$, then we have devoided the dynamic equation 3.11 of its more important features and any study of the importance of the angular momentum dependence of $V(NN)$ in the solidification problem has been irreparably undermined in its credibility. For a complete description of the neutron system we must also introduce the spin variables. Equation 3.11 will then split into three sets corresponding to $S = 0$, $M_s = 0$, and $S = 1$, $M_s = \pm 1, 0$. Canuto and Chitre solved the three sets of 7, 13, and 18 coupled differential equations by including up to $l = 6$. This was found to be large enough for the system to be stable. An fcc configuration was found to be more energetically favorable than a bcc one.

The detailed results are displayed in Table 5, where we list 1. the baryonic density, 2. the energy density, 3. the nearest neighbor distance Δ, 4. the spread of the wave function α^{-1}, and 5. the energy per baryon in MeV and then the elastic constants C_{11}, C_{12}, and C_{44} in units of 10^{-36} dynes cm^{-3}. The first

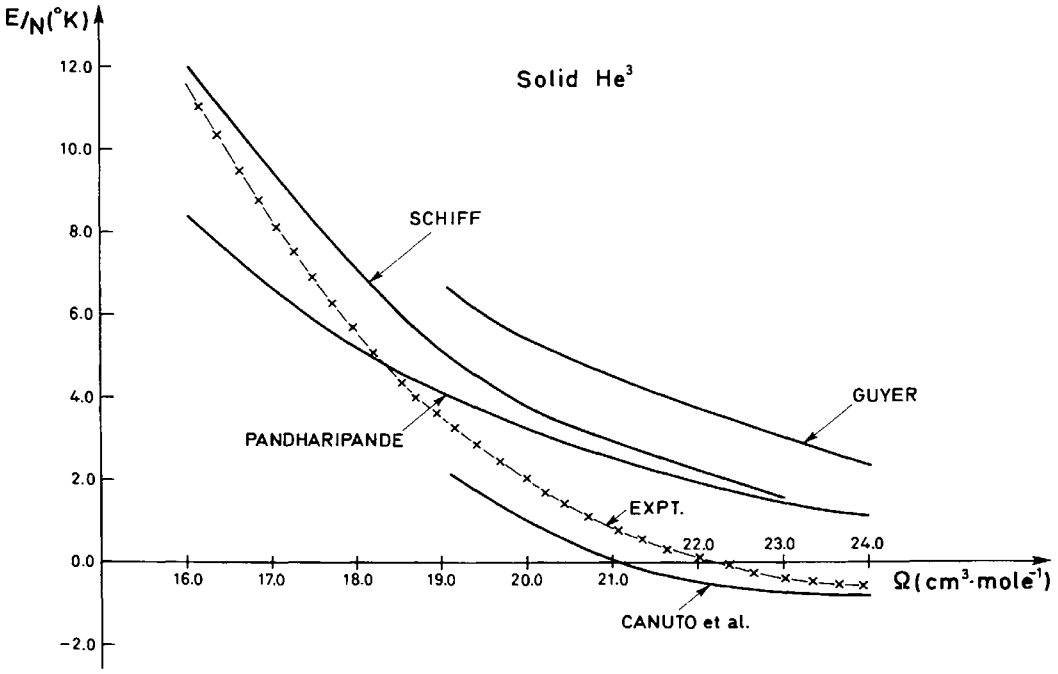


Figure 8 Ground-state energy vs molar volume for solid ^{3}He as obtained by Canuto et al (1974a). The results of Guyer (1969), Schiff (1973), and Pandharipande (1973) are reported for comparison.

Table 5 Results of the computation of V. Canuto and S. M. Chitre (1973, 1974)

n_B (fm^{-3})	$\rho \cdot 10^{15}$ (g cm^{-3})	Δ (fm)	α^{-1} (fm)	$E(n_B)$ (MeV)	C_{11}	C_{12} (10^{-36} dynes cm^{-2})	C_{44}
0.84	1.4	—	—	—	0.16	0.15	−0.03
0.96	1.6	—	—	—	0.40	0.24	0.03
1.09	1.83	1.089	0.342	163	0.89	0.37	0.09
1.44	2.4	0.995	0.312	216	2.65	1.07	0.48
2.0	3.34	0.891	0.278	322	6.78	2.97	1.47
2.63	4.4	0.817	0.254	500	17.59	6.69	3.43
2.99	5.0	0.779	0.238	612	27.57	10.33	5.71

important observation is that the oscillation around a lattice site, α^{-1}, is only $\sim 30\%$ of the first neighbor distance, Δ. In view of empirical laws like the Lindenman melting rule, there is no probability of such an oscillation producing an instability and making the crystal disappear.

The stability against elastic deformation is studied by analyzing the behavior of the elastic constants vs density, as shown in Table 5. It can be seen that the shear modulus C_{44} is positive only for $\rho \approx 1.4\ 10^{15}$ g cm^{-3}, indicating that the fcc structure of neutrons is actually a solid, that is, it can withstand shearing stresses for densities higher than this value.

3.2.6 THE WORK OF SCHIFF (1973) Another way of studying the solidification problem was devised by Schiff, who applied an idea of Wu & Feenberg (1962). At relatively high density, the major feature of a system of nucleons is undoubtedly the strong repulsion and probably not the Fermi statistics. One can therefore treat the Pauli principle as a perturbation to a Bose system, for which it is hoped that one can compute the ground state energy to good accuracy. The hypothetical boson system is studied in two steps:

1. The repulsion is considered to be represented by hard spheres. Kalos, Levesque & Verlet (1974) numerically solved the Schrödinger equation for a system of 256 hard-sphere bosons. The results are in excellent agreement with the previous results obtained using a Jastrow-type of wave function (Hansen et al 1971)
2. The second step is to treat the attractive part of the two-body potentials by first-order perturbation theory. The hard-sphere ground-state energy is lowered by up to 20%. The most complicated part of the problem consists in taking into account the Pauli principle (PP), and this is done following the prescription of Wu and Feenberg

The solid phase is treated analogously, except that the corrections due to the statistics are neglected. The final result is that a solid structure does indeed occur at a density of $(2.9 \pm 0.5) \times 10^{15}$ g cm^{-3} and a pressure of $(4.7 \pm 1)10^{30}$ atms.

If this method is applied to solid ^{3}He, one obtains the results presented in Figure 8. Even though they are not as good as those of Canuto et al (1974a),

still they have the very enviable feature of getting better as the density increases. This is indeed very pleasing, even though it could be misleadingly interpreted as a mark of merit for the method when applied to a neutron system. Such is indeed not the case, because the background hard-core bosons that produce such good results for ^{3}He cannot be considered as an appropriate basis for neutrons, whose strong interaction is far from being a hard-core.

3.2.7 THE WORK OF NOSANOW & PARISH (1974) A different approach to the solidification problem was adopted by Nosanow and Parish, who employed a Monte Carlo technique to calculate the many-body effects of the short-range correlations. The problem is formulated within the framework of the variational approach, and we have on several occasions stressed the difficulties encountered in extending such a method to cope with the angular-momentum dependence of nuclear forces. The best one can do is to use a simplified force taken to be a superposition of a singlet ($^1S_0$) and triplet potential $V_c(^3P_2)$. As in the case of Coldwell's work, the best feature of the work of Nosanow and Parish is that they treat both liquid and solid with the same many-body technique. From the methodological point of view, this is a most welcome feature.

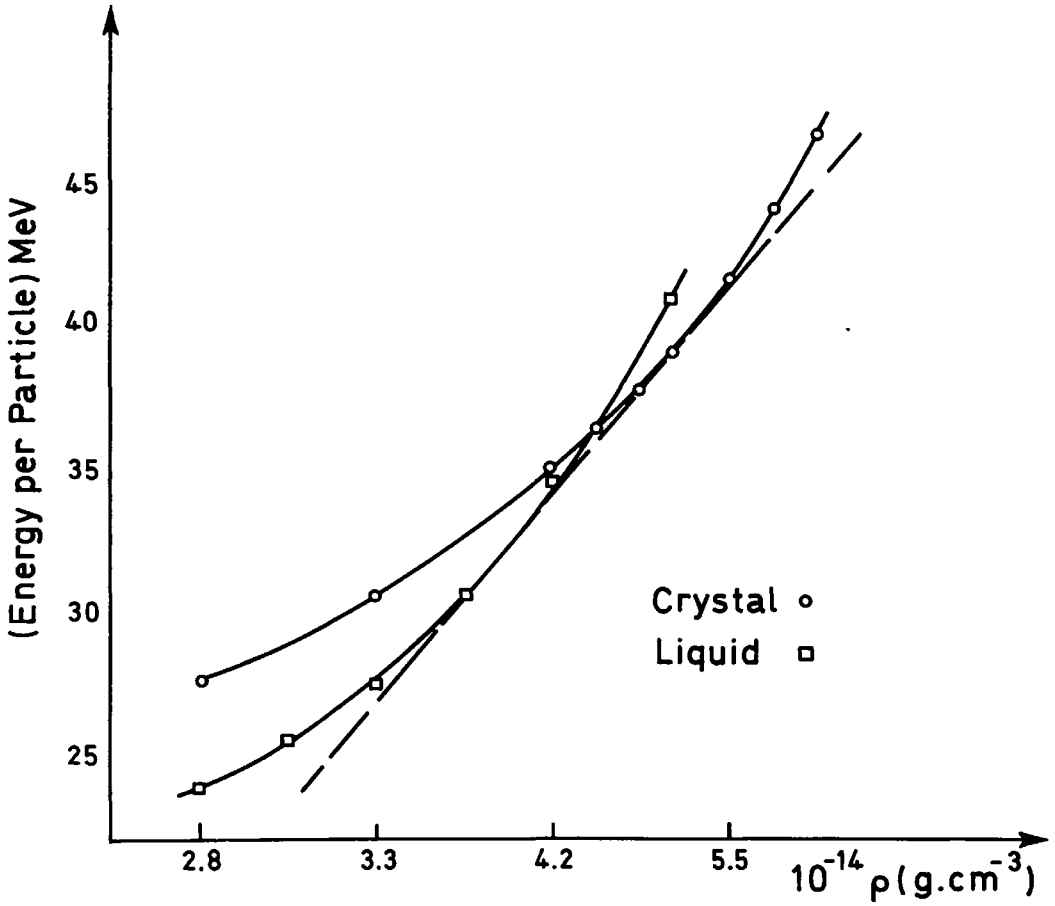


Figure 9 Liquid-solid transition in a neutron liquid as from Nosanow & Parish (1974).

The results of their computation are shown in Figure 9, where the liquid and solid energies are plotted against density. The liquid-solid transition occurs around 4.4×10^{14} g cm^{-3}; the solidification pressure is $\sim 10^{34}$ dynes cm^{-2}. These numbers are in good agreement with the ones obtained by Anderson and Palmer and by Clark and Chao but are much smaller than the ones obtained by Canuto and Chitre and by Schiff.

Nosanow and Parish do not give any indication of how sensitive the results are to the choice of potentials. Evidently there is more than one way to construct a singlet and triplet potential, and the sensitivity of the results to the amount of attraction could and should be checked. Such an analysis is still missing. More recently the same method failed to find solidification when the potential was chosen to be purely repulsive (Canuto et al 1974b). This has led us to believe that the treatment has not been fully investigated from the numerical point of view.

3.2.8 THE WORK OF PANDHARIPANDE (1973) A variational approach to the solidification problem was employed by Pandharipande (1973) who, following van Kampen (1961), expanded the ground-state energy E in clusters, truncated at the second order

$$E = \tfrac{3}{4}\hbar\omega + \tfrac{1}{2}\sum_{ij} C_2(ij). \qquad 3.12$$

The term C_2 contains three parts. The first two do not involve the potential and they almost cancel each other. The author requires that the third part, called I_3, be minimized. In this way one obtains a differential equation for ψ of the form

$$\left[-\frac{\hbar^2}{m}\nabla^2 + \tfrac{1}{4}m\omega^2(\mathbf{r}-\boldsymbol{\Delta})^2 + V(r)\right]\psi(\mathbf{r}) = \varepsilon\psi(\mathbf{r}). \qquad 3.13$$

This is precisely equation 3.11 in the Canuto and Chitre formalism. The basic idea of the computation consists in imposing reasonable restrictions on the correlation function, f, $\psi = \phi f$, so as to make the truncated expansion reasonable. Such a method is called LOCV. In the next step, 3.13 is changed into an equation for f, by filtering ϕ to the left.

We have already discussed how difficult it is to evaluate equation 3.11 because of the strong angular-momentum dependence contained in the $\mathbf{r}\cdot\boldsymbol{\Delta}$ term and in $V(r)$. The physical angular momentum is the one that characterizes ψ; by splitting ψ into ϕf, one introduces two unphysical angular momenta l_1, l_2:

$$\psi(\mathbf{r}) = \sum_l \psi_l(r) Y_l = \sum_{l_1} \phi_{l_1} Y_{l_1} \sum_{l_2} \phi_{l_2} Y_{l_2}.$$

Because both quantities $V(r)$ and $\mathbf{r}\cdot\boldsymbol{\Delta}$ are operators in the angular-momentum space that act on ψ_l, their action on either Y_{l_1} or Y_{l_2} has no meaning. This in turn implies that one cannot transport $\phi(\mathbf{r})$ to the left of 3.13, to get an equation for f. When the LOCV is applied to the test case of solid ^{3}He, Pandharipande obtains the results shown in Figure 8. The energies are much less satisfactory than those of Canuto et al (1974a). Schiff's results get better as the density increases,

whereas Pandharipande's get worse. This is rather surprising in view of the fact that the basic merit of LOCV was precisely the ability to handle the high density regime.

Having learned from the ^{3}He case the importance of the solid state term $\mathbf{r}\cdot\boldsymbol{\Delta}$, we must doubt the validity of the subsequent application of LOCV to the investigation of the importance of the l-dependence of $V(NN)$ in the solidification problem. For this reason we believe that the meaningful part of Table 6, where Pandharipande explores the importance of the l-dependence of nuclear forces, is only in the first and second case, where there is one potential for all partial waves, either purely repulsive or slightly attractive. Case 3 cannot be considered definitive in that the energies are too close. Case 4 is clearly in favor of a liquid instead of a solid structure, however.

Even if the physical idea is correct, that is, that the presence of too much attraction will favor the liquid, it is our opinion, based on the previously presented arguments and experimental facts on ^{3}He, that Pandharipande's (1973) way of handling the dynamic equation is not convincing.

3.3 *Critical Review of the Work on the Solidification Problem*

The results of the calculations so far performed are presented in Figure 10. Even supposing that all of them are correct, the spread in the reported solidification densities is still too large. Several considerations have to be made. First, not all the computations have the same degree of reliability and just looking at the various results without having this in mind can be highly misleading. Second, none of the computations have yet been repeated by any other group to check the results. An independent check, especially of the more microscopic detailed computations, would be highly desirable.

The results of Nosanow and Parish are inscrutable in that they are almost totally numerical and, since the present author is far from being an expert on Monte Carlo methods, one cannot but hope that the authors would check time and again the sensitivity of their results to the possible numerical pitfalls. The results based on the law of corresponding states, after a pleasant and even convincing

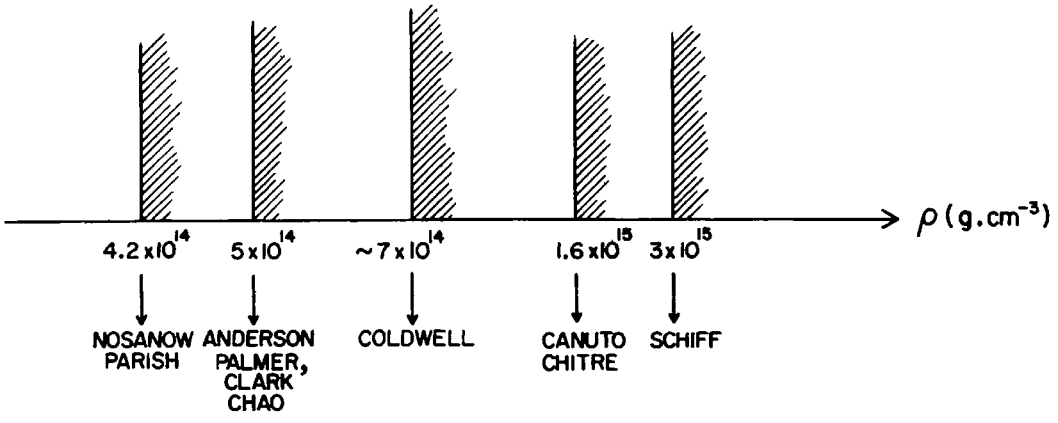


Figure 10 Present status of the results on the solidification problem.

start, face the serious problem of being too severely linked to the possibility of representing $V(NN)$ in a Lennard-Jones fashion. Even supposing that the core of $V(NN)$ is hard enough (which it is probably not) to be represented by a LJ form, still its fundamental feature of being state dependent cannot be accounted for by a simple analytic form. Despite several clever tricks devised by many people, the idea seems to have exhausted its fruitfulness.

Coldwell's results are more difficult to esteem. In fact his claim that the lack of a correlation function at short distances equally affects the solid and liquid, so as to make the results correct on a relative basis, is difficult to check.

Logically the next step is to perform a microscopic calculation. One way of dealing with that problem is due to Schiff. Here the impression is that a system of hard spheres is probably not a good representation and that in assuming such a system one actually introduces too much repulsion. One can always argue that the same amount of repulsion is also present in the liquid and therefore it should not matter very much. Even so, one cannot help feeling that if the system that Schiff has studied indeed solidifies, it has little to do with a real assembly of neutrons, whose repulsion is not, as far as we know, a hard-core. Here too, there is no way to improve upon the computation unless one has the corresponding solution of 256 Schrödinger equations with a potential less repulsive than a hard-core. This would render Schiff's treatment much more realistic.

This leaves us with only two microscopic computations by Canuto and Chitre on one hand and Pandharipande on the other. Canuto and Chitre based their reason for choosing the t-matrix on 1. the excellent results obtained for ^{3}He (Figure 8); 2. the smallness of the parameter κ, indicating the relative importance of high-order corrections; and 3. the possibility of dealing exactly with the state dependence of $V(NN)$. On the other hand, Pandharipande preferred the variational method (LOCV) to deal with points 1 and 2 because he believed he would have a better way of handling the high-density region. We have already stressed the poor performance of the LOCV method for liquids (Figure 7) and solids (Figure 8) in the high-density region.

In addition to the purely many-body aspect of the problem, one must add the series of approximations made in the case of the solid. In fact, had Pandharipande solved the dynamic equation correctly, the comparison of the results would have been very significant. As shown in Table 6, Pandharipande's contention is that by letting the potentials go from purely replusive to more realistic, the solid structure is not preferred over the liquid.

We have already stressed how this statement is not fully convincing; it is unfortunately based on an equation of motion that has, from the very beginning, been devoided of its angular-momentum features.

In the only two instances in which that equation of motion can be used with confidence, the solid actually exists (Table 6, first two cases).

On the other hand, in Canuto and Chitre's calculation, there is one unsatisfactory feature, whose implications have not been fully understood. The term $\mathbf{r}\cdot\mathbf{\Delta}$, being parity-violating, couples l with $l\pm1$, that is, brings into the problem odd waves that should not be present. At high densities, the potentials of those waves are

repulsive and therefore help solidification, so to speak. Such help should not be there, and only when a method is devised to eliminate such a spurious effect will the role of the attraction be ascertained.

In a still unpublished work, R. A. Guyer has succeeded in eliminating the unwanted waves. When he deals with a purely repulsive potential, equation 2.9, his energies are in good agreement with those of Canuto, Lodenquai & Chitre (1974b), as are those of Chakravarty et al (1974). Unfortunately, they all disagree with those of Pandharipande (column 1, Table 6), indicating that even for a unique state-independent potential, the LOCV method gives too high an energy for the solid. This is in line with what was shown in Figure 7, where the LOCV produced too high an energy for the liquid.

In conclusion, the whole problem seems to have to wait until the odd-waves contribution in Canuto and Chitre's treatment has been removed in order to check the trend of Table 6, that is, that the softness of Reid's potential prevents solidification. At the same time, within the variational framework it seems clear that the LOCV has shown its limitation in both liquid and solid phase and that a more reliable method has to be worked out.

3.4 *The Equation of State in the Solid Region*

Due to the still uncertain nature of the solid core and the rather small changes that its existence would imply in the relation $P = P(\varepsilon)$, we feel that one can safely treat this region as a liquid.

4 THE REGION $\rho > 8 \times 10^{15}$ g cm^{-3}

4.1 *Generalities*

From a judicious appraisal of the difficulties encountered in the two previous sections, we must derive a very clear if somewhat negative message. Once we exceed a density 10 times the nuclear density, the behavior of matter can no longer be described by the two conventional tools: 1. nonrelativistic many-body theories and 2. the concept of a potential.

The inclusion of hyperons offered the possibility of keeping the energy safely nonrelativistic and this prompted the extension of nonrelativistic many-body theories to the hyperonic region. The hope did not last long, because the lack of detailed knowledge of the hyperonic potentials seriously undermined the reliability of the results. Even assuming that hyperons can be used to keep the energy non-relativistic, it is clear that even the most trustworthy many-body methods cannot be stretched beyond 10 times nuclear densities. What about the *NN* potential? Even before one reaches densities at which the concept of a potential itself breaks down, we are already facing another problem. Different potentials, equally reliable on the basis of the phase-shift fitting, give rise to unpleasantly different ground-state energies. This is a simple manifestation of the fact that the region around 10^{15} g cm^{-3} is extremely sensitive to the hard-core region that is left untouched by the phase-shifts criterion. In addition to this strictly technical question, the concept of static or quasistatic *NN* potential is bound to break down. As the density increases, the meson

Table 6 Energy per particle (MeV) vs density (fm^{-3}) for different types of NN potentials

n_B (fm^{-3})	Boltzman Statistics				Fermi Statistics			
	Solid	Liquid	Solid	Liquid	Solid	Liquid	Solid	Liquid
0.4	245.6	229	—	—	—	—	—	—
0.6	408.2	399.8	—	—	—	—	—	—
0.8	589.3	594.5	—	—	—	—	—	—
1.0	788	810.5	—	—	—	—	—	—
1.2	1002	1043	370.9	337.3	364.1	334.8	—	—
1.6	1461	1543	545.7	526.7	536.8	508.6	—	—
2.0	1961	2085	738.8	736.3	729.0	702.0	—	—
2.4	2494	2660	948.1	959.6	939.3	909.7	802.9	697.8
2.8	3054	3263	1172	1204	1165	1137	1010	893.4
3.2	3637	3890	1410	1456	1405	1373	1232	1105.2
3.6	4240	4527	1656	1720	1656	1621	1467	1332.8
	$V(r) = 6484.2\,e^{-7x}/x$		$V(r) = \frac{1}{4}[V(^1D_2)+3V(^3P_2)]$		$V(r) = V(^1D_2)$ $S=0$		$V(r) = V(^1S_0)$ $S=0$, $l=0$	
					$V(r) = V_c(^3P_2)$ $S=1$		$V(r) = V(^1D_2)$ $S=0$, $l \geq 2$	
	$x = 0.7r$						$V(r) = V_c(^3P_2)$ $S=1$	

degrees of freedom cannot be eliminated in favor of a NN potential. They must be explicitly taken into account.

All these considerations call for a relativistic treatment of a many-particle system. The work that has been published in this area and that we are about to review is still of an exploratory nature. This is not a point of demerit. In a density region such as the one we are about to describe, the unknowns are so many and the pitfalls so frequent and unexpected that any type of incursion is useful, if for nothing else but to eliminate a few cases from the list of possible alternatives.

4.2 *Relativistic Hadronic Lagrangian*

Let us consider a system of nucleons interacting via scalar and vector mesons. We shall omit the pseudoscalar π, because at high density it cannot play a significant role. The scalar interaction is a simulation of a 2π system that dominates the NN interaction at intermediate densities. Even if unnecessary for most of the presentation to follow, we will think of the vector meson as the ω meson. This identification will become useful when we need the appropriate coupling constant.

The Lagrangian describing such a system is ($\hbar = c = 1$)

$$\mathcal{L} = -\bar{\psi}(\gamma_\mu \partial_\mu + M)\psi - \tfrac{1}{2}(m_s^2\phi^2 + \phi_\mu^2) - \tfrac{1}{2}m_v^2 A_\mu^2 - \tfrac{1}{4}F_{\mu\nu}^2 + g_v\bar{\psi}\gamma_\mu\psi A_\mu + g_s\bar{\psi}\psi\phi. \tag{4.1}$$

The first term is the Dirac Lagrangian for free fermions, the second the Lagrangian for a free scalar boson, the third and fourth the Lagrangians for a massive vector field (Proca field, $F_{\nu\mu} = A_{\mu\nu} - A_{\nu,\mu}$), whereas the fifth and the sixth terms are the interaction terms.

There seems to be little point at the moment in discussing how realistic equation 4.1

actually is. The derivation of a nucleon-nucleon potential is an art all by itself and it would be entirely out of place to get involved in any meaningful discussion of how many mesons ought to be included in a Lagrangian in order to be in a position to trust the resulting $V(NN)$. Needless to say, many more mesons than just a scalar and a vector are needed. If Lagrangian 4.1 does not contain all the necessary mesons, it surely does not leave out the most important ones, one attractive and one repulsive. We therefore consider 4.1 as a good reasonable starting point.

4.3 *Review of the Results*

4.3.1 THE MEAN-FIELD APPROXIMATION From equation 4.1 we can derive the equation of motion for the three fields ψ, ϕ, and A_μ as

$$(\gamma_\mu \partial_\mu + M - g_v \gamma_\mu A_\mu - g_s \phi)\psi = 0 \qquad 4.2$$

$$(\Box^2 - m_s^2)\phi = -g_s \bar{\psi}\psi \qquad 4.3$$

$$(\Box^2 - m_v^2)A_\mu = -g_v \bar{\psi}\gamma_\mu \psi + \partial_\nu \partial_\mu A_\nu. \qquad 4.4$$

In principle one can think of eliminating ϕ and A_μ in favor of ψ, via 4.3 and 4.4. The solution of 4.2 will then yield the nucleon spinor, ψ. If so, the Lagrangian 4.1 can be rewritten as a function of ψ only, thus providing the desired $P = P(\varepsilon)$ relation through the evaluation of the diagonal terms of the energy momentum tensor

$$T_{\mu\nu} = \mathscr{L}\,\delta_{\mu\nu} - \frac{\partial \mathscr{L}}{\partial \psi_\mu}\psi_\nu \qquad \psi_\mu \equiv \partial\psi/\partial_{x_\mu}. \qquad 4.5$$

Needless to say, such a program has no chance of being implemented unless one resorts to some kind of approximation. The method employed can be called a mean-field approximation or, alternatively, a relativistic Hartree approximation. It was first used by Marx (1956) and by Marx & Nemeth (1964) for a purely scalar field, and more recently rediscussed in the context of high-density matter by Kalman (1974). It was first employed for a pure vector interaction by Zeldovich (1962), who made an extensive study of the high-density limit of the relation $P = c_s^2 \varepsilon$, both classically and quantum mechanically, reaching the well-known conclusion that $c_s^2 \to 1$, as $\varepsilon \to \infty$, that is, at superhigh density the velocity of sound approaches the velocity of light. From now on, the question of the asymptotic behavior of c_s^2 will frequently arise and we had better prepare the reader for a series of discrepant results. Maximum stiffness corresponds to $c_s^2 \to 1$, whereas maximum softness corresponds to $c_s^2 \to 0$. For a free gas, the well-known result is $c_s^2 \to \frac{1}{3}$. In the opinion of the author, the balance is presently tilted in favor of $c_s^2 \to 1$, even though it is very hard to find data, astrophysical or otherwise, through which to settle the question unequivocally. The more extensive study of 4.1 for both scalar and vector interaction has been performed by Walecka (1974); we review this work.

In the spirit of the mean-field approximation, we perform an average on both sides of 4.3 and 4.4, by substituting ϕ and A_μ with $\langle\phi\rangle$ and $\langle A_4\rangle$. The solution of 4.3 and 4.4 is then trivial. Upon substituting $\langle\phi\rangle$ and $\langle A_4\rangle$ in 4.2, we obtain a new Dirac equation in which both the mass and the energy get renormalized. Once

Dirac's equation is solved, ψ is substituted back into 4.3 and 4.4, thus providing the self-consistent solutions for $\langle\phi\rangle$ and $\langle A_4\rangle$. The equation of state is then readily obtained. We will not write down the analytic expression, equations 3.51 and 3.56 of Walecka's paper. By inspecting the $P = P(\varepsilon)$ relation, it is easy to show that as $\varepsilon \to \infty$, the contribution of the scalar meson is negligible compared to the vector meson. In the same limit $\varepsilon \to \infty$, Walecka recovers the Zeldovich result:

$$P = c_s^2 \varepsilon, \qquad c_s^2 \to 1. \tag{4.6}$$

The numerical values for Walecka's $P = P(\varepsilon)$ are presented in Table 7.

Even though several points remain to be studied, for example, the stability of the ground state against fluctuations and the inclusions of more mesons, the idea of a

Table 7 Equation of state for a pure neutron gas after Walecka (1974)

n_B (fm^{-3})	$\rho = \varepsilon/c^2$ ($\mathrm{g\,cm}^{-3}$)	P ($\mathrm{dynes\,cm}^{-2}$)	$P/\rho c^2$
0.70	1.149×10^{15}	2.829×10^{35}	0.273
0.80	1.367×10^{15}	4.066×10^{35}	0.330
0.90	1.604×10^{15}	5.469×10^{35}	0.379
1.0	1.859×10^{15}	7.034×10^{35}	0.420
1.1	2.132×10^{15}	8.756×10^{35}	0.456
1.25	2.576×10^{15}	1.163×10^{36}	0.501
1.4	3.060×10^{15}	1.484×10^{36}	0.539
1.6	3.768×10^{15}	1.964×10^{36}	0.579
1.8	4.547×10^{15}	2.503×10^{36}	0.611
2.0	5.397×10^{15}	3.100×10^{36}	0.638
2.2	6.315×10^{15}	3.755×10^{36}	0.661
2.4	7.303×10^{15}	4.468×10^{36}	0.680
2.6	8.360×10^{15}	5.239×10^{36}	0.696
2.8	9.485×10^{15}	6.067×10^{36}	0.711
3.0	1.068×10^{16}	6.952×10^{36}	0.723
3.2	1.194×10^{16}	7.894×10^{36}	0.735
3.4	1.327×10^{16}	8.893×10^{36}	0.745
3.6	1.466×10^{16}	9.948×10^{36}	0.754
3.8	1.612×10^{16}	1.106×10^{37}	0.762
4.0	1.765×10^{16}	1.223×10^{37}	0.770
4.2	1.925×10^{16}	1.345×10^{37}	0.777
4.4	2.091×10^{16}	1.474×10^{37}	0.783
4.6	2.264×10^{16}	1.607×10^{37}	0.790
4.8	2.443×10^{16}	1.747×10^{37}	0.794
5.0	2.629×10^{16}	1.892×10^{37}	0.800
5.2	2.822×10^{16}	2.043×10^{37}	0.804
5.4	3.021×10^{16}	2.199×10^{37}	0.809
5.6	3.226×10^{16}	2.361×10^{37}	0.813
5.8	3.438×10^{16}	2.529×10^{37}	0.817
6.0	3.657×10^{16}	2.702×10^{37}	0.821

constant mesonic field at superhigh density seems a fruitful one. The advantage of this model, when compared with the ones based on the concept of a $V(NN)$ potential, is that it is not a perturbative approach with the coupling constant as a smallness parameter. In a restricted sense, the coupling constant is treated exactly to all orders of perturbation theory. From the many-body point of view, clearly only the lowest-order many-body diagrams have been included. For example, neither the nucleon self-energy nor the polarization tensor in the equation of motion for the mesons are included in this formalism.

4.3.2 THE PERTURBATIVE APPROACH An effort in this direction has been recently undertaken by Zimmerman and collaborators (Bowers & Zimmerman 1973a,b, Bowers, Campbell & Zimmerman 1973a–c). From the relativistic many-body theory it is known that the pressure and energy density can be computed if one knows the two-point Green's function, solution of the Dyson equation

$$(\gamma_\mu \partial_\mu + M + \Sigma)G = 1. \tag{4.7}$$

The general expression for Σ requires the knowledge of G itself and the exact meson Green's functions, say $\mathscr{D}$. In turn $\mathscr{D}$ is given as a function of the polarization tensor, Π, whose computation again requires the knowledge of G. This formidable chain of self-consistent equations is approximated by the authors at the first stage, that is, in the evaluation of Σ, by identifying $\mathscr{D}$ and G with the corresponding free-particle Green's functions, thus obtaining for Σ

$$\Sigma = ig^2 \int \frac{d^4k}{(2\pi)^4} \gamma G^{(0)} \gamma \mathscr{D}^{(0)}. \tag{4.8}$$

Given the exploratory nature of this analysis, this series of approximations seems natural; if the incentive is great enough, one could look for the changes brought in by the inclusion of more terms. However, this is probably not the most significant part. In fact, a more serious difficulty is represented by the treatment of the coupling constant only up to the second order. In this respect, the present treatment faces the same kind of criticism as those employing a NN potential. A convergence analysis would imply the evaluation of not just one, but several terms in powers of g^2, with the almost inevitable result that such a series does in fact not converge under the present circumstances. In our opinion this is the weakest and most difficult to amend feature of the whole treatment.

The explicit evaluation of 4.8 is carried out by taking $\gamma \equiv \gamma_5$, that is, by supposing that the nuclear force is mediated by pions. Unfortunately, this is not a very realistic choice for the high-density regime, where the pions will clearly play a negligible role and the vector mesons will undoubtedly dominate, as we have seen explicitly in Walecka's work. It turns out that in the high-density regime the present computations predict $c_s^2 \to \frac{1}{3}$. This has given rise to some misunderstandings about the behavior of the velocity of sound as $\varepsilon \to \infty$ (Ruffini 1973, Cameron & Canuto 1973). Although we have no quarrel with the relation obtained by Zimmerman and collaborators, we would like to stress that it cannot be taken as a proof that c_s^2 actually approaches $\frac{1}{3}$ in the real world. The exchange of pions is a good representa-

tion of the low-density regime and the extrapolation of the resulting $P = P(\varepsilon)$ relation to the high-density regime has only an academic interest. We do not know with any certainty the behavior of c_s^2 as the density increases. The whole point, however, is that the present calculation cannot be used in any meaningful way in that respect.

If, instead of a pion, the authors had exchanged a vector meson, the comparison with Walecka's results would have been more instructive. Due to the general nature of the Zeldovich argument, we feel that the relativistic treatment of Zimmerman and collaborators is bound to recover the relation $c_s^2 \to 1$, when the vector meson is introduced. It is indeed hard to believe that the introduction of the self-energy operator could change the density dependence in such a way as to upset the $c_s^2 \to 1$ relation. For the sake of completeness, we reproduce below the $P = P(\varepsilon)$ relation obtained by Zimmerman and collaborators, even though it can be used meaningfully only in the low-density regime.

$$P = P_0 x^4 [1 - 24x^{-4} + 4\, e^{-x} x^{-4}(6 + 6x + 3x^2 + x^3)]$$
$$\varepsilon = \varepsilon_0 [9/4(8 + x^4) - 9\, e^{-x}(2 + 2x + x^2) - 2x^3] \qquad 4.9$$
$$P_0 = \frac{729}{500} \frac{M^4 c^5}{\pi^2 \hbar^3} \qquad \varepsilon_0 = \tfrac{4}{3} P_0 \qquad x = \frac{5}{9} \frac{\hbar k_F}{Mc}.$$

4.3.3 THE HAGEDORN-TYPE FORMULATIONS The two previous investigations, though different in their conception and technical aspects, nevertheless have several features in common. The work we are about to review departs radically from any field theoretical consideration and the results are therefore more difficult to compare with anything we have already presented.

Even supposing that we could clear up all the problems that still plague the two previous treatments, there always remains the possibility that we should actually go beyond the basic baryonic octet and include the whole host of baryonic resonances that comprise about 354 baryons up to 2.2 GeV. We do not actually know if such excited states ought to be included or not, but it is clear that should the answer be positive, none of the previous formulations would be adequate for such a job.

We review in what follows a series of papers that have dealt with this problem by postulating that such a host of baryonic states should indeed be accounted for because it actually determines the behavior of the $P = P(\varepsilon)$ relation at superhigh densities. We also present a serious criticism that has caused some misgivings about the validity of the models.

The simplest way to evaluate the equation of state for a system of excited baryons is to suppose that the constituents are free, but with their masses adjusted to account for the interaction. Sawyer (1972) first noticed that the masses of the resonances that we read from the tables refer to free decaying resonances and may be significantly different from the ones we are dealing with in a dense medium. Since the mass shift is actually unknown, Leung & Wang (1973) postulated that the effective mass spectrum due to the mass shifts obeys a power law, that is, that the number of baryons below a certain mass m can be represented as

$$N(m) = Am^{2a}. \qquad 4.10$$

The Equation of state for a free ensemble of baryons with such a mass spectrum is easily found with the result

$$P = c_s^2 \varepsilon, \qquad c_s^2 = (3+2a)^{-1}. \tag{4.11}$$

If we postulate that the parameter a is the same as the one obtained by fitting equation 4.10 to the experimentally known masses, then $a = 2.9$ and $c_s^2 \to 0.11$, an extremely low value, if compared with the previous values of $\frac{1}{3}$ or 1.

Leung and Wang's work is a particular case of a more general line of approach due to Hagedorn (1970), who proposed a quasifree or asymptotically free model for baryonic matter by pointing out that, at very high densities, the interaction may be (partially?) accounted for by precisely including the whole spectrum of baryonic states. The point being made is that the existence of the baryonic states is per se a manifestation of the interaction and therefore instead of trying the impossible job of taking into account the interactions, one just counts the existing baryons, establishes a mass spectrum of the form 4.10, or more complicated (Hagedorn 1970), and then treats the system as free (Frautschi et al 1971, Wheeler 1971). The idea is very simple and very appealing. A general feature common to all the results obtained with the previous method, or modification of it, is that the resulting equation of state is very soft. We quote here an expression due to Wheeler (1971) and Hagedorn (unpublished):

$$\begin{aligned} P &= \rho c^2 [\ln (\rho/\rho_0)]^{-1} \\ \rho_0 &= 2.5\ 10^{12}\ \text{g cm}^{-3}. \end{aligned} \tag{4.12}$$

For $\rho \to \infty$, the velocity of sound goes to zero, contrary to $c_s^2 \to 1$, derived before. How can we assess the value of such theories? We discuss in the next section the possible asymptotic values of c_s^2 and what the experimental data seem to indicate. For the time being, we would like to present a theoretical criticism to the previous way of computing $P = P(\varepsilon)$, due to Sawyer (1972).

In one way or the other, the previous approaches rely on the existing data on excited baryonic states to fix the parameters of the supposed mass spectrum. This is clearly true in the polynomial fit of Leung and Wang, and even if it is not trivially clear in the Hagedorn type of approach, where the parameters of the more complicated spectrum

$$N(m) = m^{\alpha} \exp(\beta m) \tag{4.13}$$

are derived theoretically, it is a common practice to show that such behavior at least does not contradict the existing data. The main point is that the data in one way or another are actually used. Sawyer has made the valid point, however, that such masses refer to free decaying baryons, whereas the ones we are interested in concern particles imbedded in a dense medium. We said before that the Hagedorn formulation can be rephrased by saying that the chemical potential of one particle

$$\mu = (m^2 + p^2)^{1/2} + U(p) \tag{4.14}$$

can actually be rewritten as

$$\mu = (m^{*2} + p^2)^{1/2}, \tag{4.15}$$

that is, the two-body interaction can be absorbed in an effective mass. Sawyer's point is that the most important contribution to the effective mass is not actually originating from the two-body potentials, but from the mass shift that the dense surrounding medium exerts on that particle. Sawyer computed the mass shift by evaluating the self-energy operator Σ (see equation 4.7) for a Δ^- (1236 MeV) immersed in a neutron medium. The energy shift increases with density and goes from $+140$ to $+400$ MeV in the density region $0.5 \leqq n_B \leqq 5$ (fm^{-3}), even in the absence of a two-body interaction.

The effect is of a general nature and is therefore expected to hold for any other type of resonance. If so, two consequences are immediately obvious:

1. Because each resonance is actually heavier than we thought, it will require a higher density to have it come in and all the hyperonic configurations presented before can therefore be drastically changed. For instance, the appearance of Δ^- at 1.5×10^{15} g cm^{-3} (Figure 6) can actually be quenched. The same argument would apply to all the previous computations
2. The baryonic-level density formula, as used previously, seems now quite unlikely. In fact the energy shift can become so large that no bound state beyond the basic baryonic state is ever populated for any density. If so, the equation of state would be actually much harder than the one proposed by the Hagedorn type of formalism

4.4 *Concluding Remarks on the High-Density Region*

We can only concur with Sawyer that his computation has actually shown, if not the incorrectness of treating the baryonic resonances as a free system with the mass spectrum given by the experimental values, at least the strong need for a much more careful examination of the treatment. In view of this serious difficulty, we conservatively regard Walecka's computation as the most adequate and reliable way, presently known, of describing the region $\rho \geqq 8 \times 10^{15}$ g cm^{-3}.

5 THE RELATION $P = c_s^2 \varepsilon$ IN THE LIMIT $\varepsilon \to \infty$

5.1 *Generalities*

We concluded Section 2 on the hyperonic liquid suggesting that the most reliable $P = P(\varepsilon)$ relation in the interval $2 \times 10^{14} \leqq \rho \leqq 7.7 \times 10^{15}$ g cm^{-3} is the one provided by Bethe and Johnson. We concluded Section 4 indicating how Walecka's work can be taken to continue the previous equation of state from 8×10^{15} g cm^{-3} upwards. If so, we can consider concluded the problem of the high-density behavior of matter. The feeling remains, however, that in deciding among the various possibilities, we should have presented what the experimental data, astrophysical or otherwise, suggest, instead of using theoretical arguments only.

We have intentionally used theoretical arguments only because the experimental data have been, until very recently, of little or no use at all. The situation has now somewhat changed and we describe in what follows the way one can use the high-energy data on p-p collisions in a meaningful way.

5.2 *Neutron Stars*

Despite the fact that neutron stars are indeed the denser objects presently available in astrophysics, they are not dense enough to allow us to ascertain the behavior of matter at densities 10 times higher than nuclear density. Rhoades & Ruffini (1971) computed the mass and radius of a stable neutron star by using a very hard ($c_s^2 \to 1$) and a very soft ($c_s^2 \to 0$) equation of state starting at 10 times the nuclear density. The two possibilities are indicated in Figure 12.

The largely expected result was that neither quantity was sensitive to the high-density behavior of P vs ρ. Because neither mass nor radius can be measured, one often employs the moment of inertia. In the popular dipole model for the Crab nebula it is thought that the energy loss is optical and X-ray synchrotron radiation in the amount of (Baldwin 1971)

$$0.2 \times 10^{38} \left(\frac{\text{ergs}}{\text{sec}}\right) \frac{1}{(\text{kpc})^2}$$

must be replenished by the loss of rotational energy $I\Omega\dot{\Omega}$. Because the distance of the Crab pulsar is quite uncertain and it can be anywhere between 1.25 and 2.5 kpc (Trimble & Woltjer 1971, Börner 1973), it follows that I must at least be such as to satisfy the condition

$$0.288 \times 10^{38} < I\Omega\dot{\Omega}.$$

For the Crab pulsar $P = 2\pi/\Omega = 0.033$ sec, and $P\dot{P} = 1.4 \times 10^{14}$, so that I must be at least greater than 0.62×10^{44} g cm^2. If we assume an average distance of 2 kpc, then I must be greater than 1.72×10^{44} g cm^2. From the work of Rhoades and Ruffini, the moment of inertia for the two extreme cases can be computed and found to satisfy these lower bounds. This reiterates the well-known fact that observational properties of neutron stars cannot be used to study the behavior of matter at 10 times the nuclear density.

5.3 *High-Energy* p-p *Collisions: The Hydrodynamic Model*

Canuto & Lodenquai (1975) have recently proposed that some of the properties of high-energy p-p collisions could be the most useful data available to study the behavior of the speed of sound at superhigh density. Experimentally it is known that the collision of two energetic (10^2–10^4 GeV) protons is accompanied by the production of a host of other particles whose multiplicity increases with the energy of the incoming protons.

The two incoming protons are strongly Lorentz contracted in the direction of motion, the contraction factor being M/E_c. One possible model describing the p-p collision, the so-called statistical hydrodynamical model, calls for the formation, right after the collision, of a hot compressed disc with dimensions

$$V = \frac{4\pi}{3}\left(\frac{\hbar}{m_\pi c}\right)^3 \left(\frac{2Mc^2}{E_c}\right) \qquad 5.1$$

that subsequently expands under its own pressure. The energy density in the initial

disc is easily computed to be

$$\rho = \varepsilon/c^2 = E_c/c^2 V = 1.5 \times 10^{14} E_L \text{ (g cm}^{-3}\text{)}, \quad 5.2$$

where $2E_L = E_c^2$ is the lab energy in GeV. Typically, for $E_L \sim 10^3$ GeV, $\rho \sim 10^{17}$ g cm^{-3}. The hadronic matter inside the hot disc has therefore an energy density greater than the one encountered at the center of a neutron star. It is to be expected that the development of such a hadronic matter will strongly depend on the assumed equation of state. The original model for such a hot hadronic gas given by Fermi (1950), followed by a revision due to Pomeranchuck (1951), was systematically worked out in its full mathematical complexity and implications by Landau (1953). Continuous refinements and improvements on the original Landau model have been accomplished over the years, notably by Feinberg and his school [(1965); see (1972) for a full account of the model].

The central point of the Landau model is that the original hot disc evolves in time in a way that must be described by the relativistic Navier-Stokes Equation

$$[(P+\varepsilon)U_\mu U_\nu + P\delta_{\mu\nu}]_{,\nu} = 0. \quad 5.3$$

This requires an equation of state. Landau, upon using $P = \frac{1}{3}\varepsilon$, arrived at a series of important relations that only recently, with the advent of new experimental data, have been restudied and reanalyzed. The program of many of the modern versions of the Landau model consists in solving 5.3 for a general c_s^2, whose value is then adjusted to fit the data. Needless to say, the job is difficult and several alternatives exist, but the important point is that, contrary to the neutron star case, there is a clear prospect of achieving some positive results.

Suhonen et al (1973) analyzed two types of data: the multiplicity N vs energy and the distribution of $N(\eta)$ vs η, the so-called rapidity distributions. The multiplicity was found to be

$$N \sim E^{(1-c_s^2)/(1+c_s^2)}. \quad 5.4$$

If 5.4 is correct, we must conclude that c_s^2 cannot be 1, because we know that N increases with E. The authors' contention is that c_s^2 should be close to 0.28 or $\frac{1}{3}$ in order to fit the existing data. The argument is not correct, however. In the first place, formula 5.4 is given only within a log E term, because it is derived from statistical mechanics. This alone implies that $c_s^2 = 1$ is perfectly legitimate, since the multiplicity can easily be fitted by log E, if the fitting is started at, say, 10^2 GeV, as one should, for the statistical model is certainly not applicable at lower energies. In the second place, it is not at all clear that the measured multiplicity has anything to do with the one computed from 5.4. Many intermediate processes have intervened and the prehistoric age to which 5.4 refers has been obliterated to the point of being irrelevant.

The rapidity distribution curve specifically depends upon the Landau model and the authors claim to have solved 5.3 numerically for $P = c_s^2\varepsilon$. Two sets of experimental data from the Pisa-Stony Brook collaboration have been employed and the result is that c_s^2 can be either $\frac{1}{2}$ or $\frac{1}{3}$ but not 1. We cannot express any sound opinion on this second point because we do not know how accurate the numerical

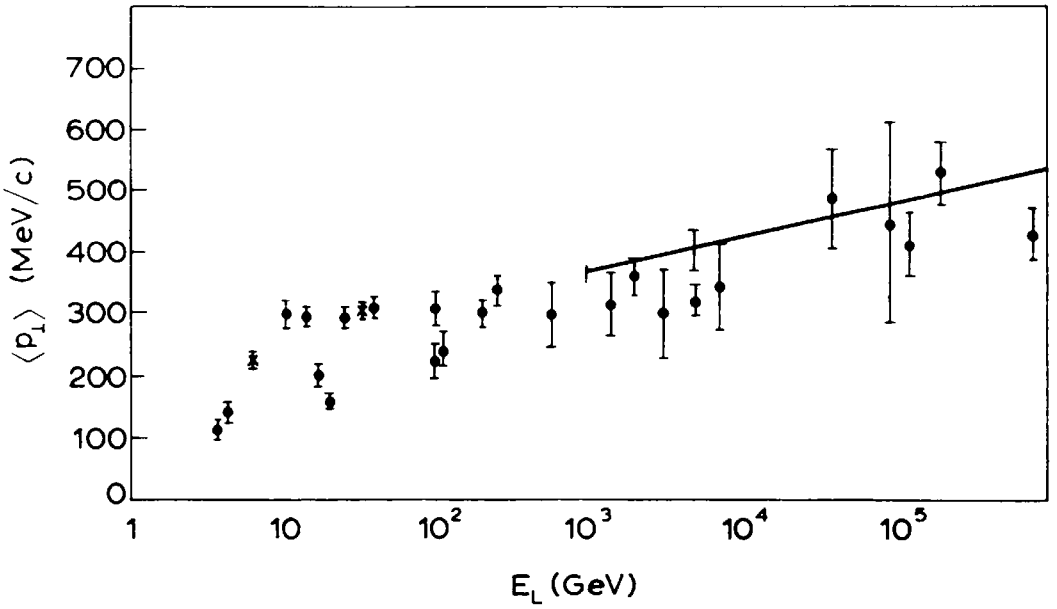


Figure 11 Transverse momentum vs E_L. The data are from Feinberg (1972). The solid line corresponds to $c_s^2 = \frac{1}{3}$.

solution actually is. The boundary conditions imposed by Landau are rather tricky and can lead to inconsistencies if not treated properly. We have to reserve judgement on this point.

In recent work by Satz and collaborators (Chaichian et al 1974) the point was made that the more reliable experimental quantity is probably the constancy of the transverse momentum with respect to the energy of the incoming protons. This fact, known for many years, prompted Cocconi (1959) to propose the existence of an ultimate temperature, an idea at the very basis of the Hagedorn model. Using the solution of the Landau model for a general c_s^2, Satz and collaborators derived the following expression for the transverse momentum

$$p_T = p_0 E^{\frac{c_s^2(1-c_s^2)}{1+c_s^2}}. \qquad 5.5$$

In Figure 11 we present the fit to several experimental points as from $c_s^2 = \frac{1}{3}$, that is, $p_T \sim E^{1/6} \sim E_L^{1/12}$. The fit is indeed excellent. Very satisfactory agreement can also be obtained by using $c_s^2 = 1$, since both 5.4 and 5.5 are given to within a

Table 8 Value of the velocity of sound (in units of c) from several pionic Lagrangians

Lagrangian	$L_0{}^a - \lambda\phi^{2n}$	$L_0 - \lambda\phi^{2n} - \nu\phi^{2k}$	$[1 - l^4(\dot{\phi}_\mu^2 - m_\pi^2\phi^2)]^{1/2}$	$\frac{\phi_\mu^2}{(1+F^{-2}\phi^2)^2} + \frac{m_\pi^2\phi^2}{(1+F^{-2}\phi^2)}$
c_s^2	$\frac{n-1}{n+1}$	$\frac{n-k}{2kn+k-n}$	0	1
Constancy of p_T	yes if $n \gg 1$	yes if $n-k \sim kn$	?	yes

[a] L_0 is the Free-pion Lagrangian.

ln E term. The value $c_s^2 = 0$ is excluded because with no pressure, there is no hydrodynamic expansion. As discussed by Canuto & Lodenquai (1975) the value $c_s^2 = 1$ is perhaps preferable since the ln E behavior so obtained is the same as the one predicted by the multiperipheral model. We therefore assume that c_s^2 indeed goes to one as the energy density goes to infinity. This being the case, one can not only have a handle on the high-density behavior of the equation of state, but also use the relation $c_s^2 \to 1$ ($\varepsilon \to \infty$) to decide among several pionic Lagrangians that have been proposed over the years. Canuto & Lodenquai (1975) analyzed four well-known cases with the results presented in Table 8.

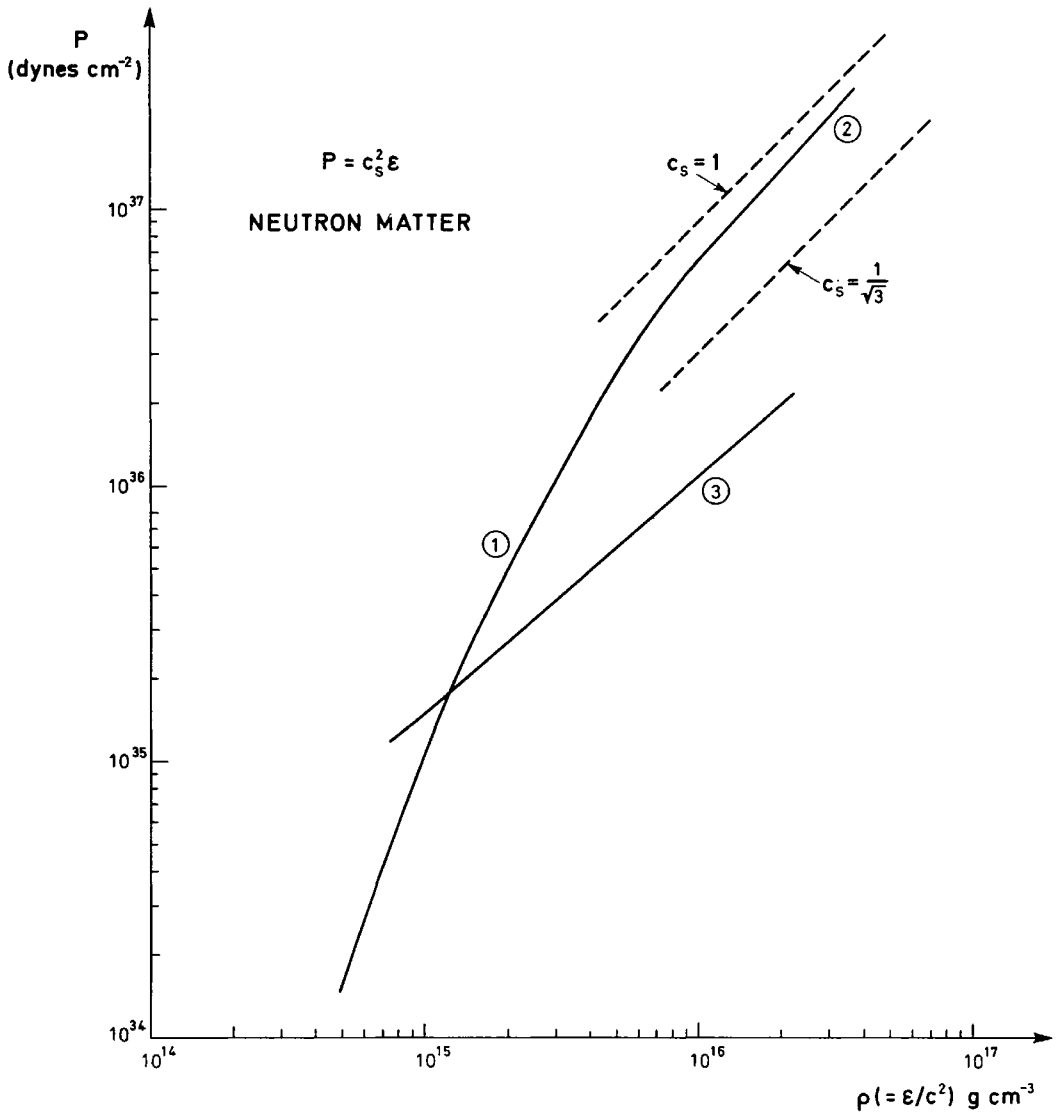


Figure 12 The "best" composite $P = P(\varepsilon)$ relation presently available. Up to 7.7×10^{15} g cm^{-3} (*curve 1*) we employ the Bethe-Johnson results. From 7.7×10^{15} g cm^{-3} (*curve 2*) we joined smoothly with the relativistic expression of $P = P(\varepsilon)$ given by Walecka (1974). Also shown are the causality limit $c_s = 1$ and the free-particle case $c_s = 1/(3)^{1/2}$. The Hagedorn-Wheeler equation of state (*curve 3*) is also shown for comparison.

The first two cases were already studied by Milekhin (1962). The third Lagrangian, originally proposed by Born & Infeld (1934), was used in this context by Heisenberg (1952). The last one has been recently proposed by Weinberg (1968). The first and second case can fit the constancy of p_T under appropriate restrictions on the indices n and k. The third one gives $c_s^2 \to 0$ and therefore no hydrodynamic expansion. The last Lagrangian satisfies the p_T test because it yields $c_s^2 \to 1$. It is the most modern and reliable Lagrangian and it is comforting that it is also in agreement with the behavior of matter at very high density.

5.4 *Conclusions*

We conclude this review by suggesting what, in our opinion, is the most reliable equation of state one can presently have covering the range of 10^4–10^{18} g cm^{-3}:

1. Up to a density of 2×10^{14} g cm^{-3}, any of the relations $P = P(\varepsilon)$ presented in Table 5(1) can be used
2. From 2×10^{14} up to 7.7×10^{15} g cm^{-3}, one can use any one of the $P = P(\varepsilon)$ relations given in Tables 10–12 of Part 1, or Table 4 of the present article. Because we do not actually know how reliable the computations that include hyperons are, it is perhaps more appropriate to regard this density regime as a neutron liquid. This will limit the choice to Tables 10 (Part 1) or 4*a*. We would suggest Table 4*a*
3. Starting at 7.7×10^{15} g cm^{-3}, we can adopt Table 7. Such a composite relation is shown in Figure 12, curves 1 and 2, together with the free-particle case ($c_s^2 = \frac{1}{3}$), the causality limit ($c_s^2 = 1$), and the Hagedorn-Wheeler equation 4.12, curve 3.

In conclusion, we would like to stress that, in spite of the impressive advances accomplished in the past few years, the problem of high-density matter cannot be considered satisfactorily understood unless further study and work is undertaken to reveal the missing facets of the hyperon liquid, the neutron solid, and the superhigh-density regime.

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